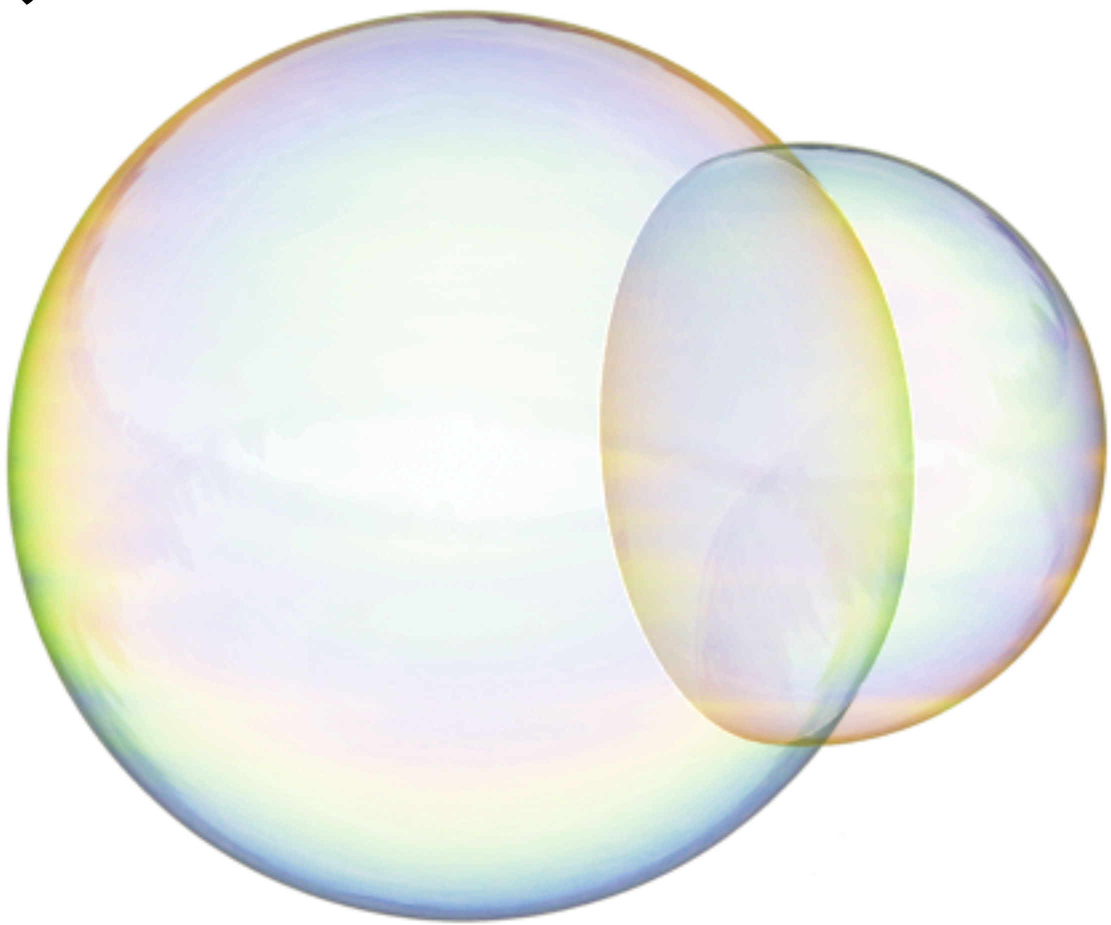
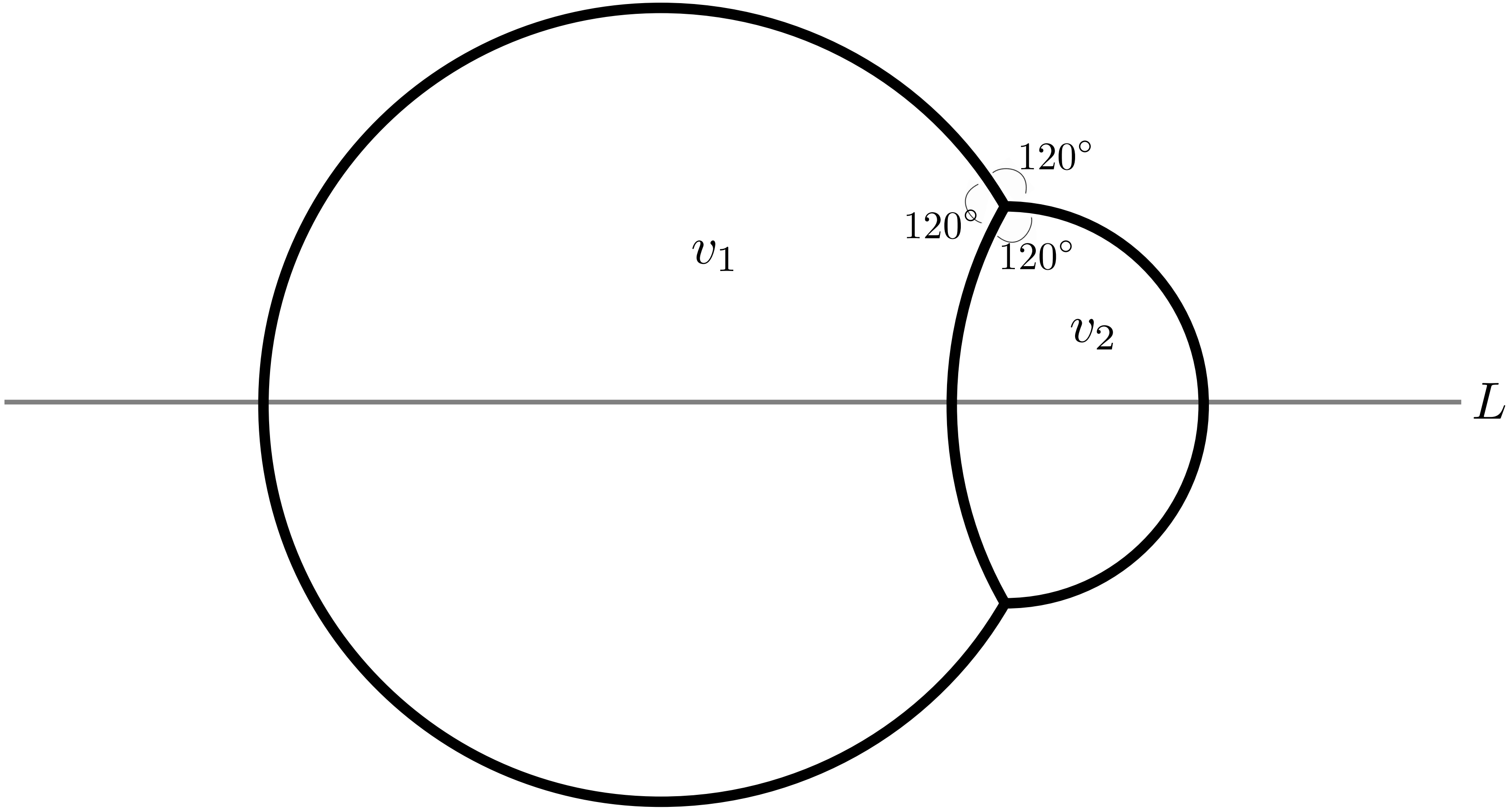
Proof of the Double Bubble Conjecture in $\mathbf{R}^n$

Ben Reichardt
Caltech



Double Bubble Theorem

- **Theorem:** The least-area way to enclose and separate two given volumes in $\mathbf{R}^n$ is the standard double bubble.



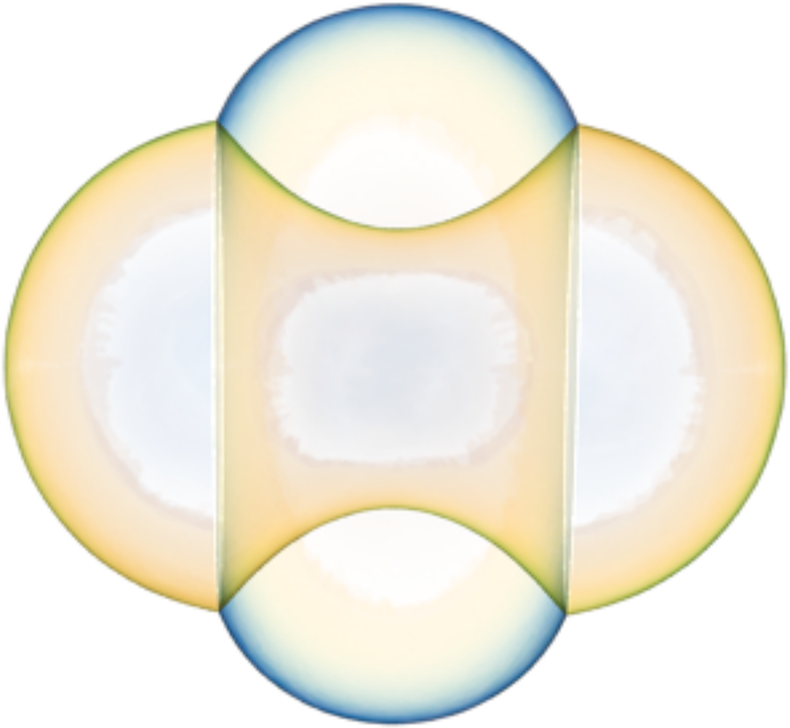
- three spherical caps centered on the axis L, meeting at 120 degree angles

History

- **Theorem:** The least-area way to enclose and separate two given volumes in $\mathbf{R}^n$ is the standard double bubble.
- Proof in $\mathbf{R}^2$ by Foisy, Alfaro, Brock, Hodges, Zimba (1993)
- Proof for equal volumes in $\mathbf{R}^3$ by Hass, Hutchings, Schlafly (1995)…

Hutchings Structure Theorem

- **Theorem** [Hutchings, 1997]: Only possible nonstandard minimizers are rotationally symmetric about an axis L, and consist of "trees" of annular bands wrapped around each other.



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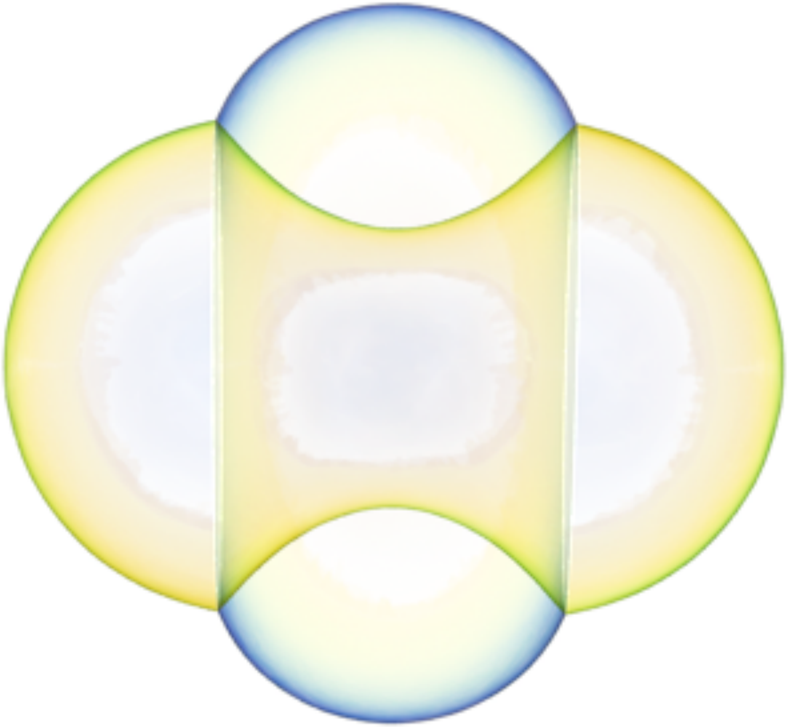

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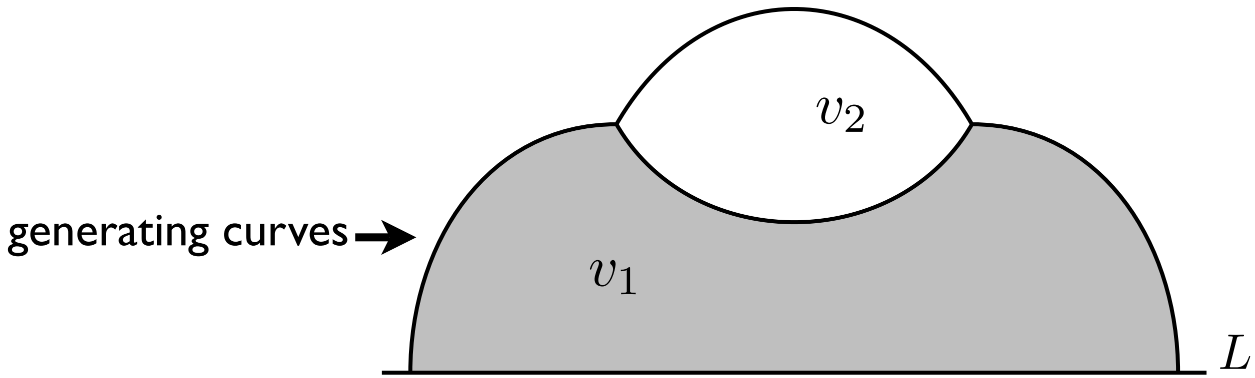
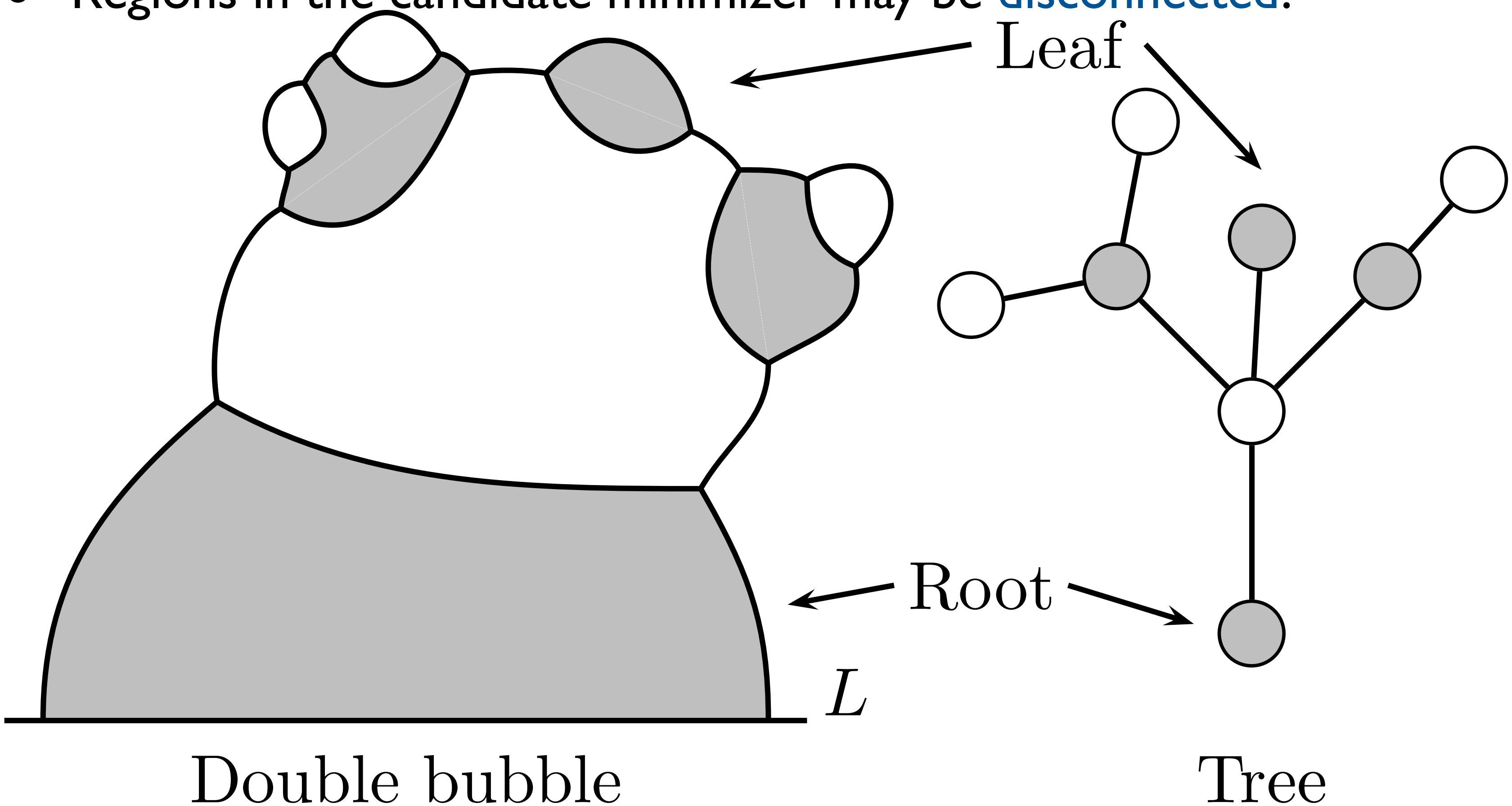


Boundaries are constant-mean-curvature surfaces meeting at 120° angles.

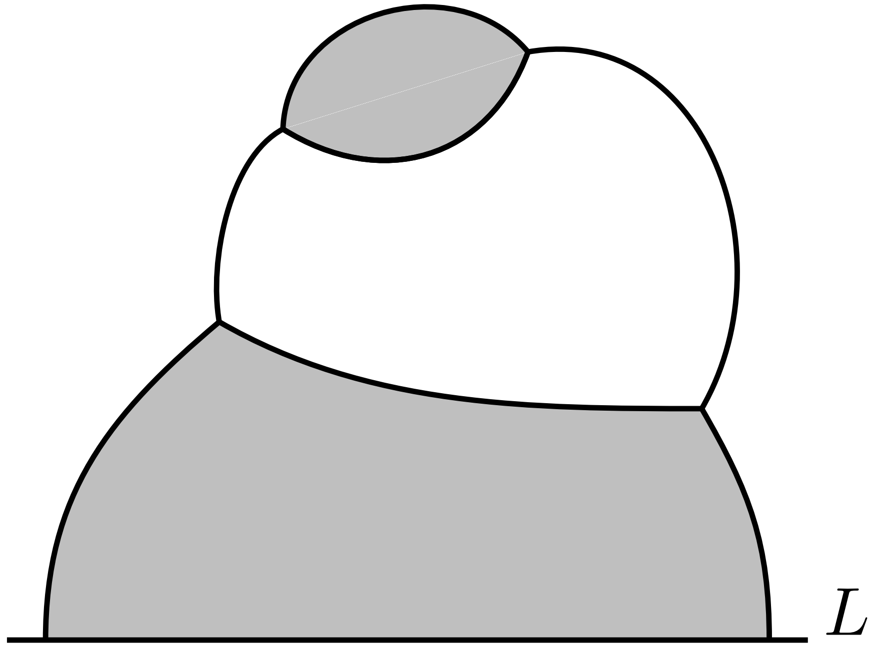
Hutchings Structure Theorem

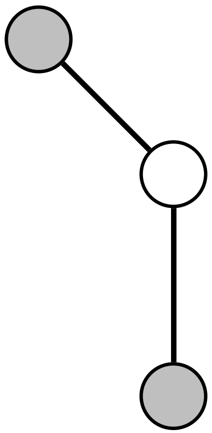
- **Theorem** [Hutchings, 1997]: Only possible nonstandard minimizers are rotationally symmetric about an axis L, and consist of "trees" of annular bands wrapped around each other.
  Boundaries are constant-mean-curvature surfaces meeting at 120° angles.
  - Regions in the candidate minimizer may be disconnected!



History—Proof in $\mathbf{R}^3$

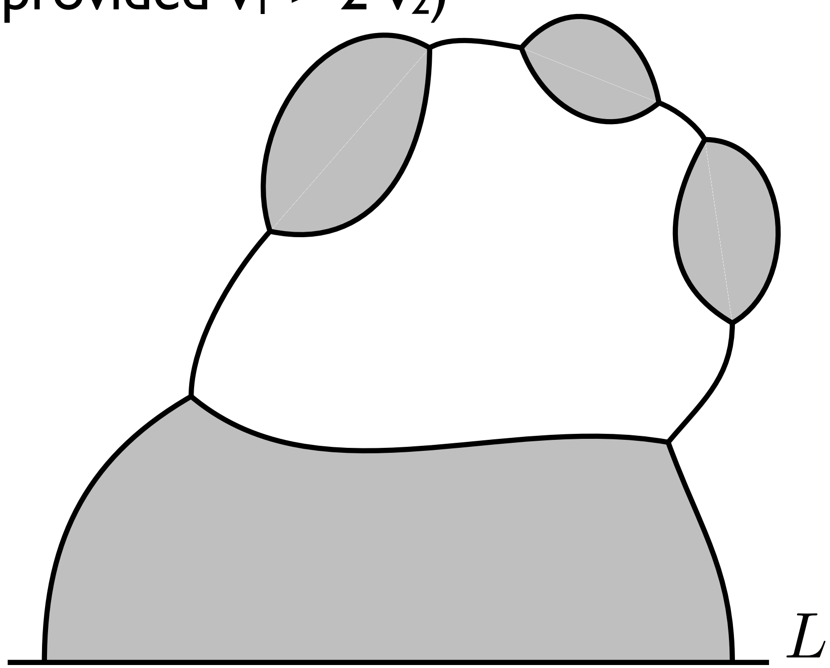
- Proof in $\mathbf{R}^2$ by Foisy, Alfaro, Brock, Hodges, Zimba (1993)
- Proof for equal volumes in $\mathbf{R}^3$ by Hass, Hutchings, Schlafly (1995)
- Proof in $\mathbf{R}^3$ by Hutchings, Morgan, Ritoré, Ros (2002)
  - Hutchings bounds ('97) guarantee that larger region is connected and smaller region has at most two components, in $\mathbf{R}^3$
  - Proof is by eliminating as unstable nonstandard "1+1" and "1+2" bubbles

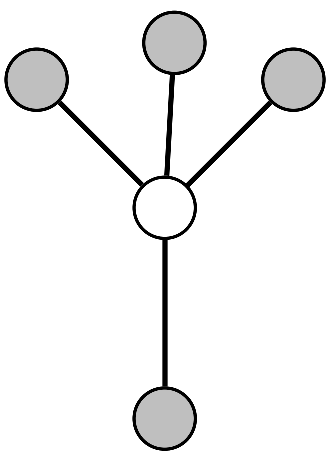




History—Proof in $\mathbf{R}^4$

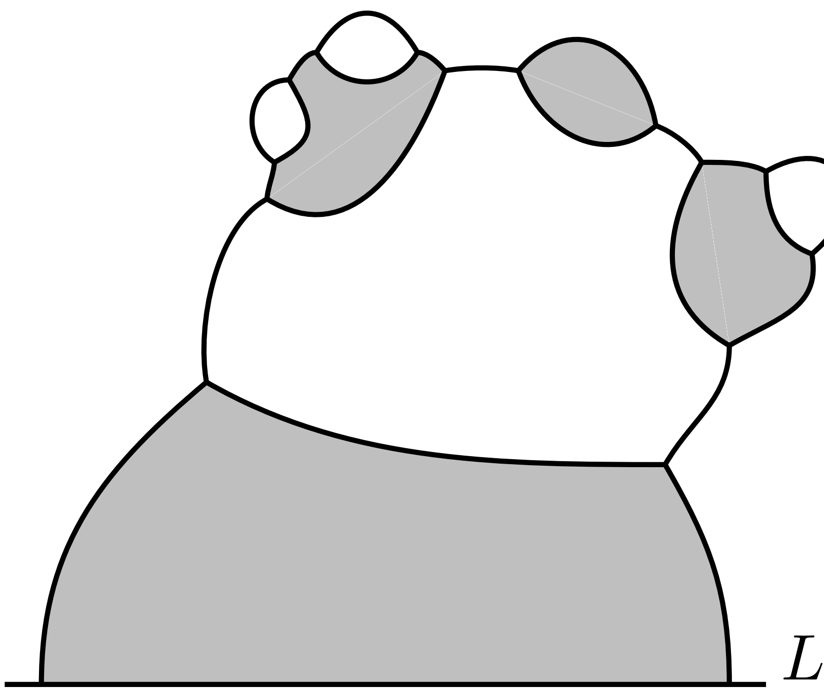
- Proof in $\mathbf{R}^2$ by Foisy, Alfaro, Brock, Hodges, Zimba (1993)
- Proof for equal volumes in $\mathbf{R}^3$ by Hass, Hutchings, Schlafly (1995)
- Proof in $\mathbf{R}^3$ by Hutchings, Morgan, Ritoré, Ros (2002)
  - by eliminating "1+1" and "1+2" bubbles (trees with up to three nodes)
- Proof in $\mathbf{R}^4$ by Reichardt, Heilmann, Lai, Spielman (2003)
  - by eliminating "1+k" bubbles—larger region is connected in $\mathbf{R}^4$ (and in $\mathbf{R}^n$ provided $v_1 > 2\,v_2$)

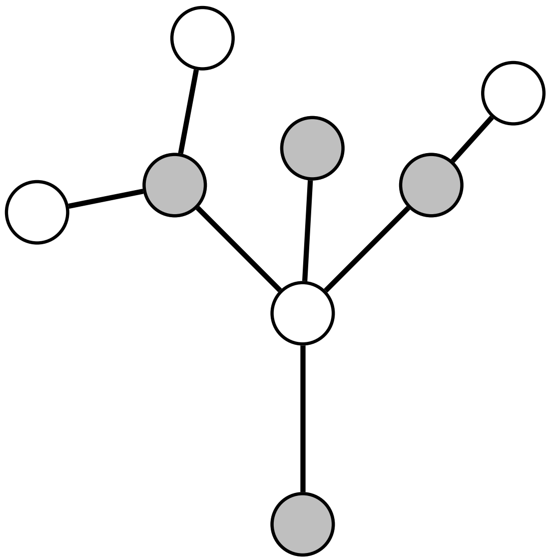




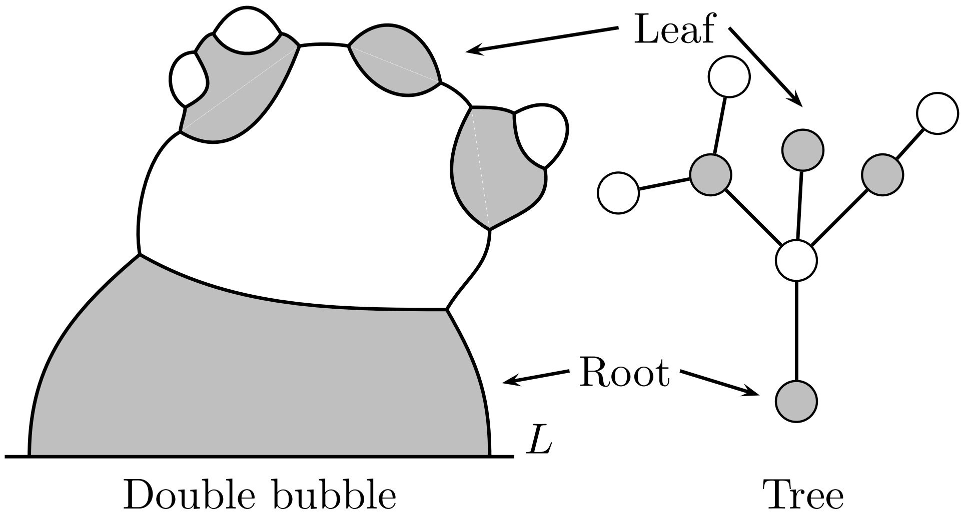
Proof in $\mathbf{R}^n$, n≥3

- Proof in $\mathbf{R}^3$ by Hutchings, Morgan, Ritoré, Ros (2002)
  - by eliminating "1+1" and "1+2" bubbles (trees with up to three nodes)
- Proof in $\mathbf{R}^4$ by Reichardt, Heilmann, Lai, Spielman (2003)
  - by eliminating "1+k" bubbles—larger region is connected in $\mathbf{R}^4$
- Proof in $\mathbf{R}^n$ is by eliminating as unstable all nonstandard "j+k" bubbles
  - component bounds, which worsen with n, aren't needed





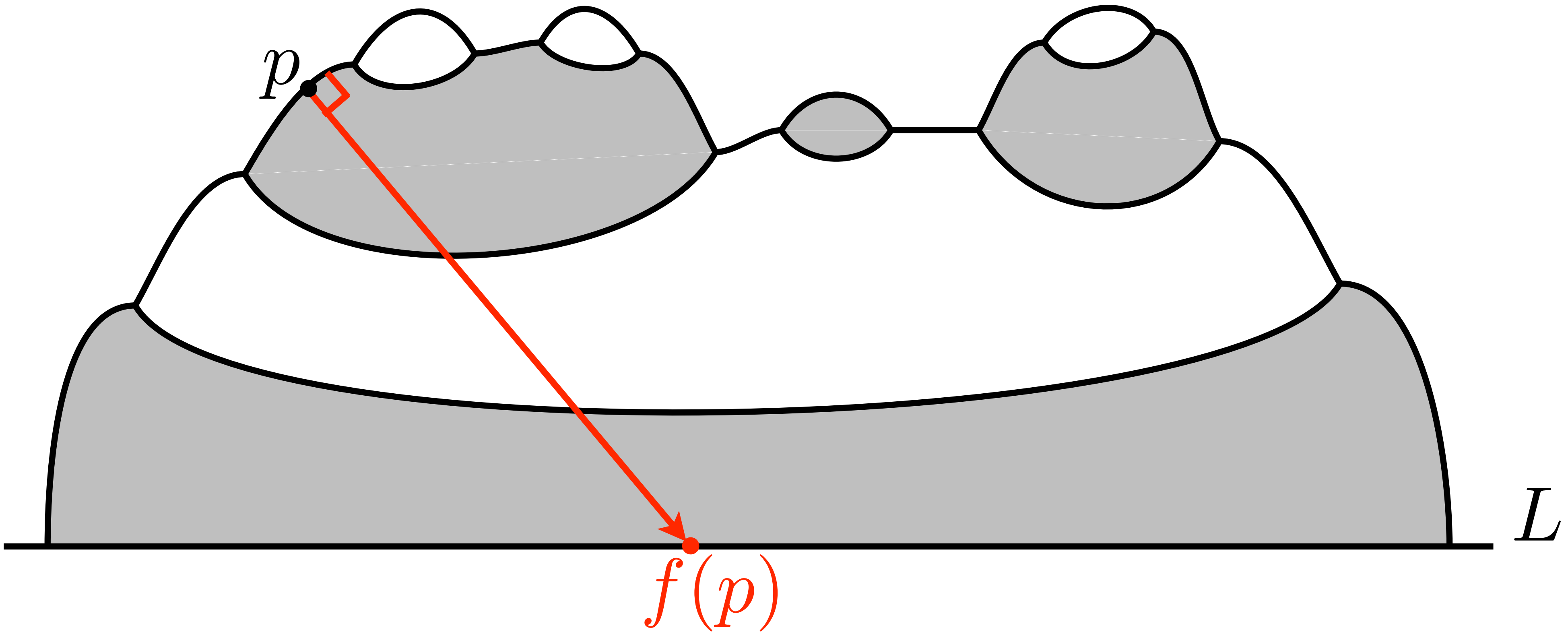
Talk sketch



- Double Bubble Theorem
- History
- Hutchings Structure Theorem
- Proof sketch
  - Instability by separation [HMRR '02]
  - Elimination of (near) graph nonstandard bubbles
  - Inductive reduction to (near) graph case

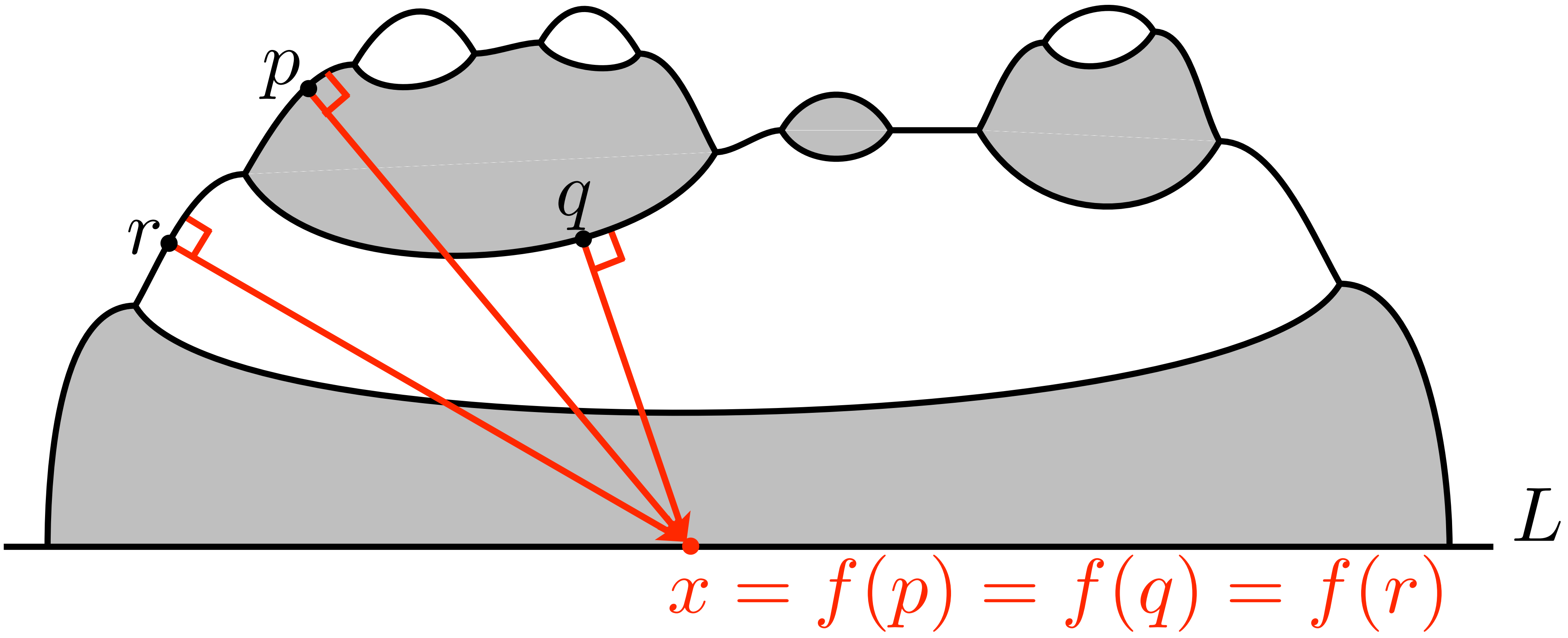
Instability by separation

- **Definition:** f: {generating curves} → L
  - extend the downward normal at p until it hits L



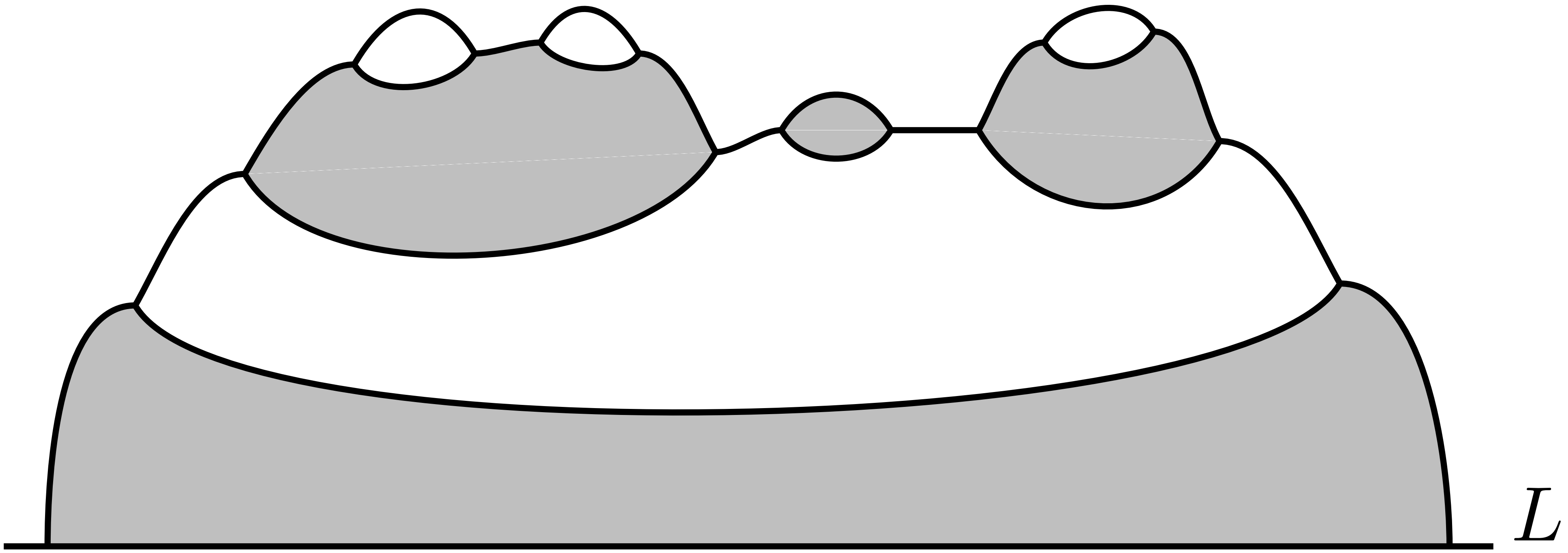
Instability by separation

- **Definition:** f: {generating curves} → L
  - extend the downward normal at p until it hits L
- **Separation Lemma** [HMRR '02]: {$f^{-1}(x)$} cannot separate the generating curves



Case of graph generating curves

- **Definition:** f: {generating curves} → L, extend downward normal to hit L
- **Separation Lemma:** {$f^{-1}(x)$} cannot separate the generating curves
- Assume that all pieces of the generating curves are graph above L (no piece turns past the vertical)—want to find a "separating set"



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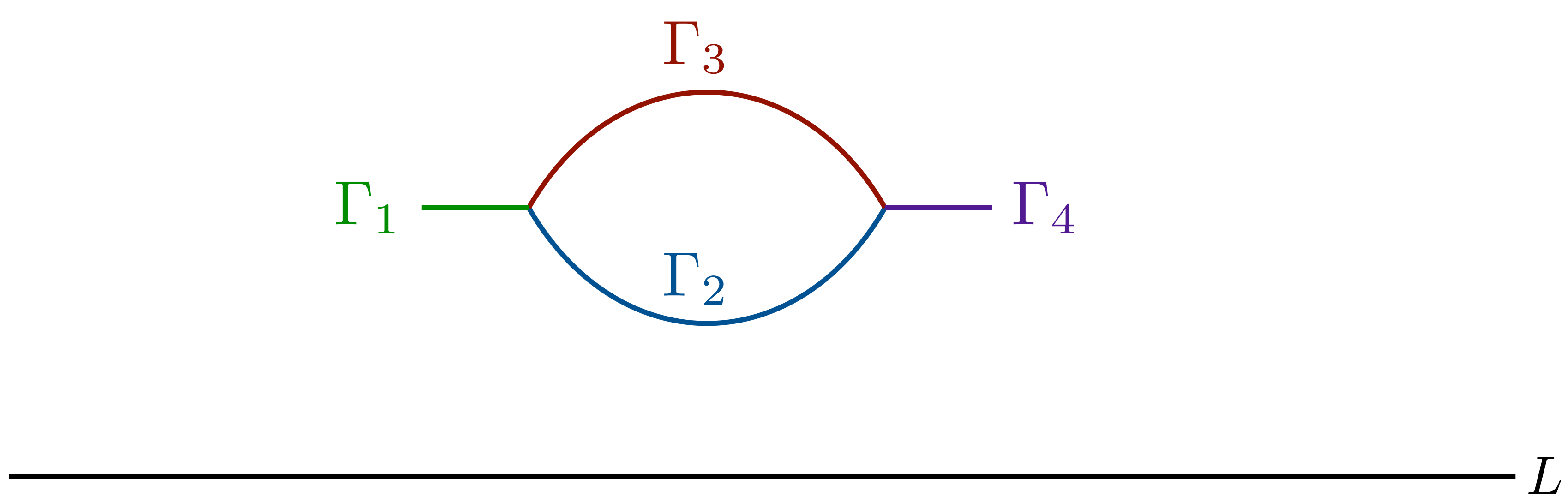
- Consider a leaf component…



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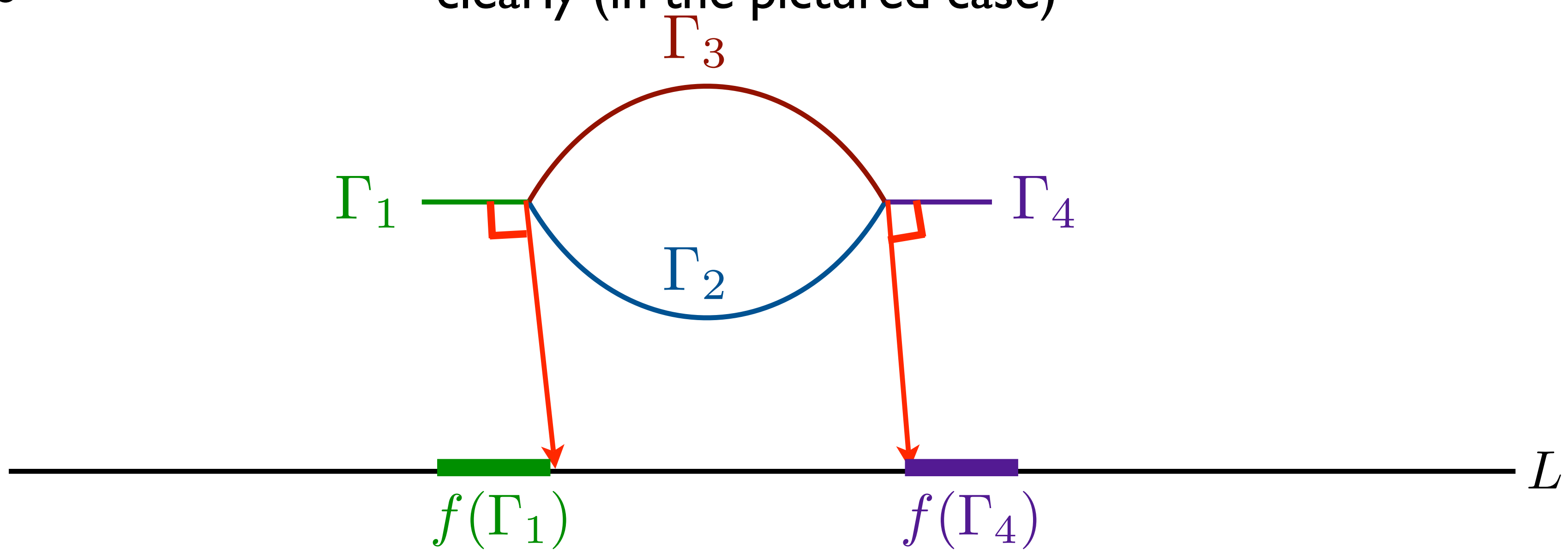
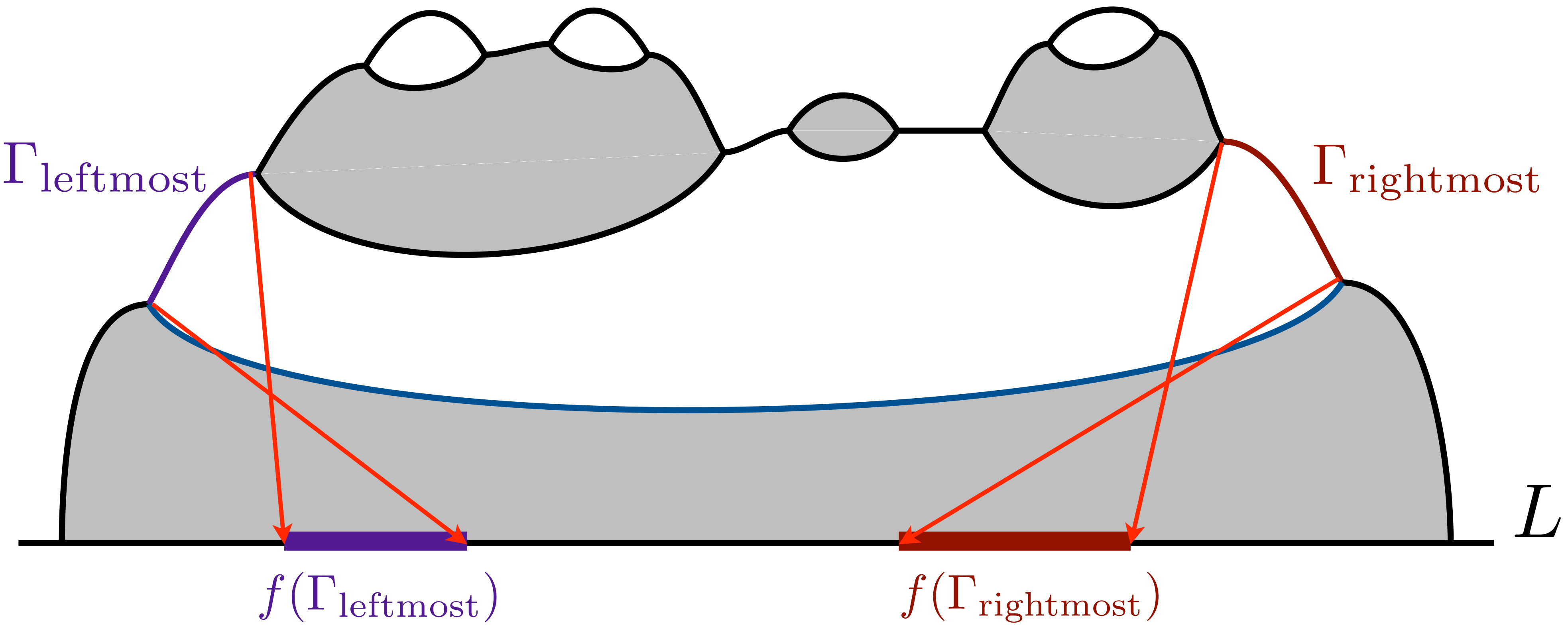
- **Separation Lemma:** {$f^{-1}(x)$} cannot separate the generating curves
- Assume that all pieces of the generating curves are graph above L (no piece turns past the vertical)—want to find a "separating set"
- Consider a leaf component…
  - Separation Lemma ⇒ $f(\Gamma_1) \cap f(\Gamma_4) = \emptyset$
  - $f(\Gamma_1) < f(\Gamma_4)$ clearly (in the pictured case)



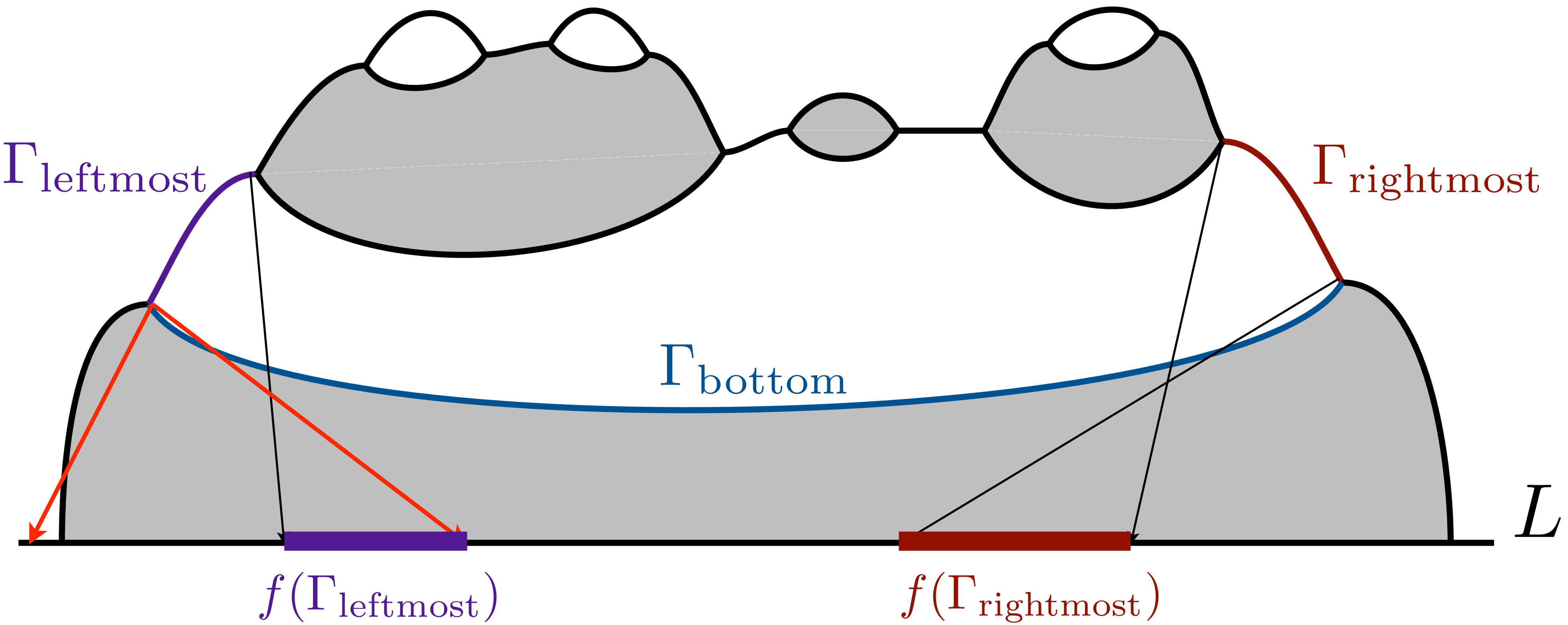
Case of graph generating curves

- **Separation Lemma:** $\{f^{-1}(x)\}$ cannot separate the generating curves
- Assume that all pieces of the generating curves are graph above L (no piece turns past the vertical)—want to find a "separating set"
- Repeating leaf argument… get $f(\Gamma_{leftmost}) < f(\Gamma_{rightmost})$



Case of graph generating curves

- **Separation Lemma:** $\{f^{-1}(x)\}$ cannot separate the generating curves
- Assume that all pieces of the generating curves are graph above L (no piece turns past the vertical)—want to find a "separating set"
- Repeating leaf argument… get $f(\Gamma_{\text{leftmost}}) < f(\Gamma_{\text{rightmost}})$
- But $f(\Gamma_{\text{bottom}})$ starts left of sup $f(\Gamma_{\text{leftmost}})$…



Case of graph generating curves

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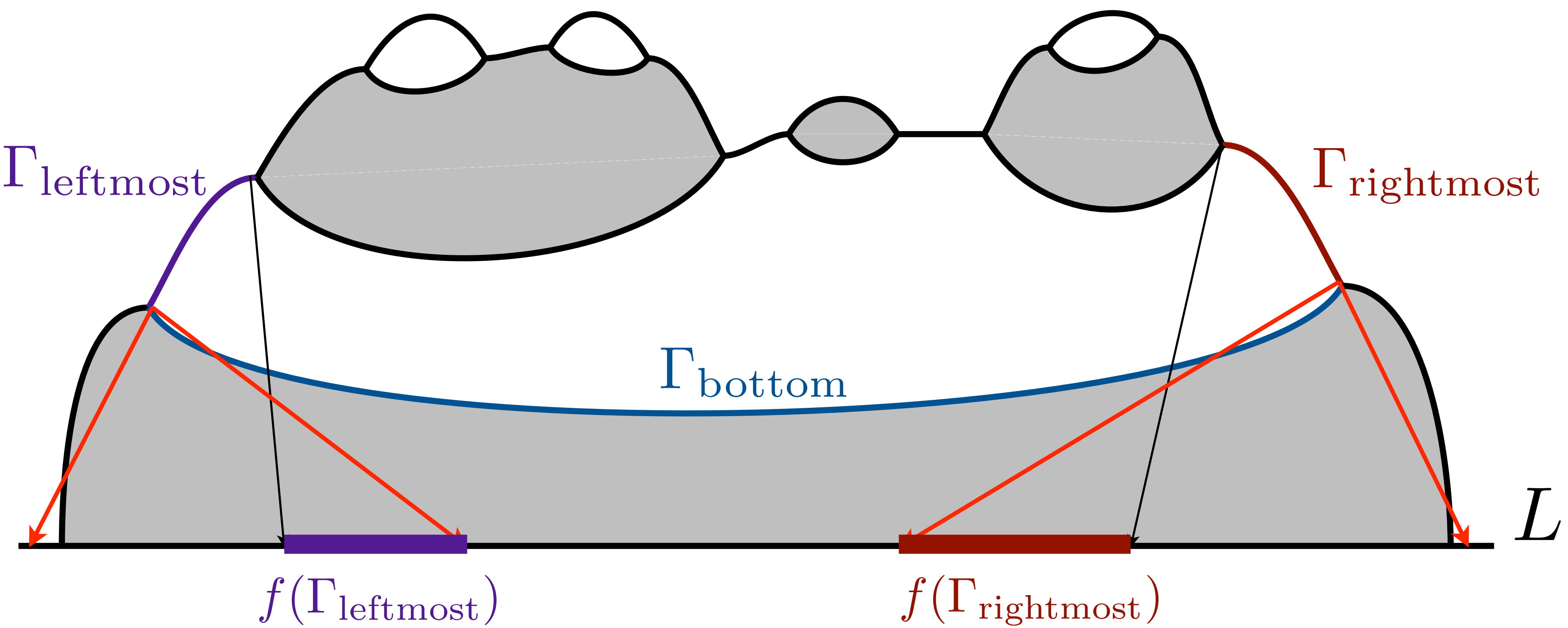

Case of graph generating curves

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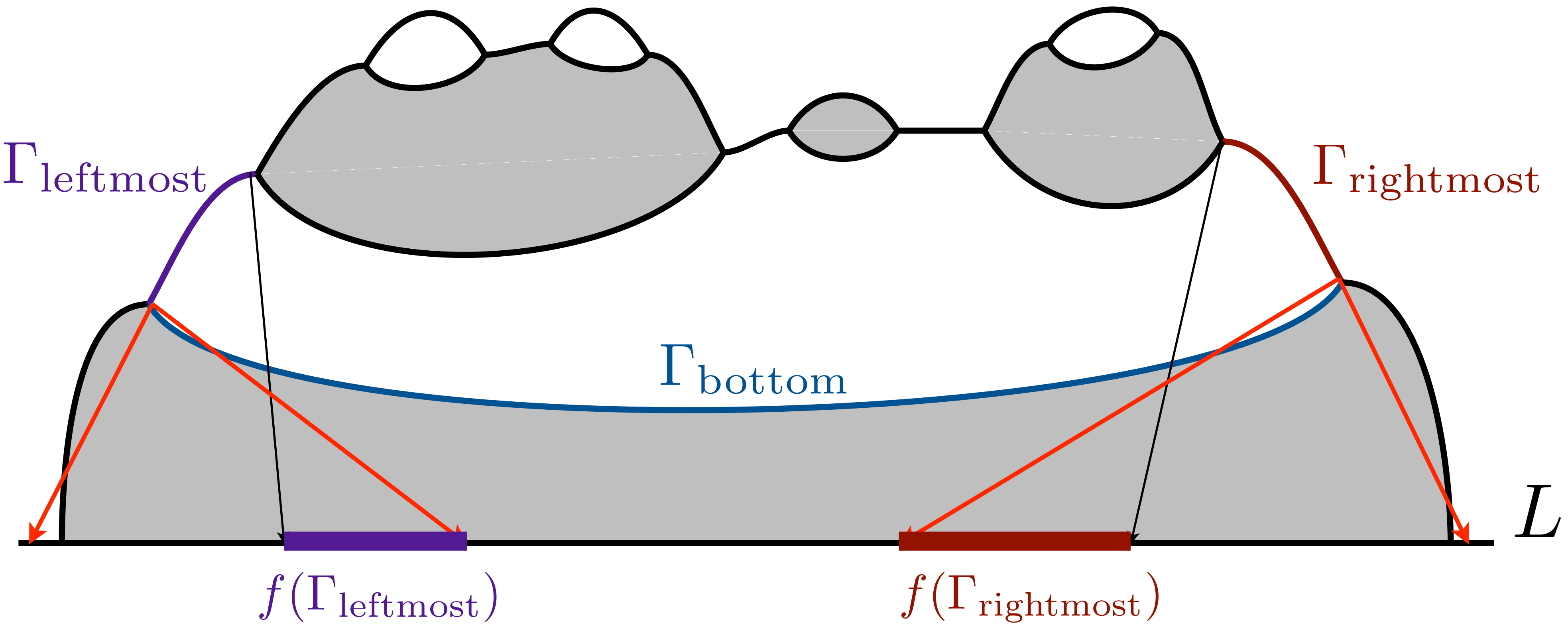
- But $f(\Gamma_{\text{bottom}})$ starts left of sup $f(\Gamma_{\text{leftmost}})$ and ends above inf $f(\Gamma_{\text{rightmost}})$

∴ There is a Γ_{bottom}, Γ_{leftmost} separating set! $(f(\Gamma_{\text{bottom}}) \cap f(\Gamma_{\text{leftmost}}) \neq \emptyset)$

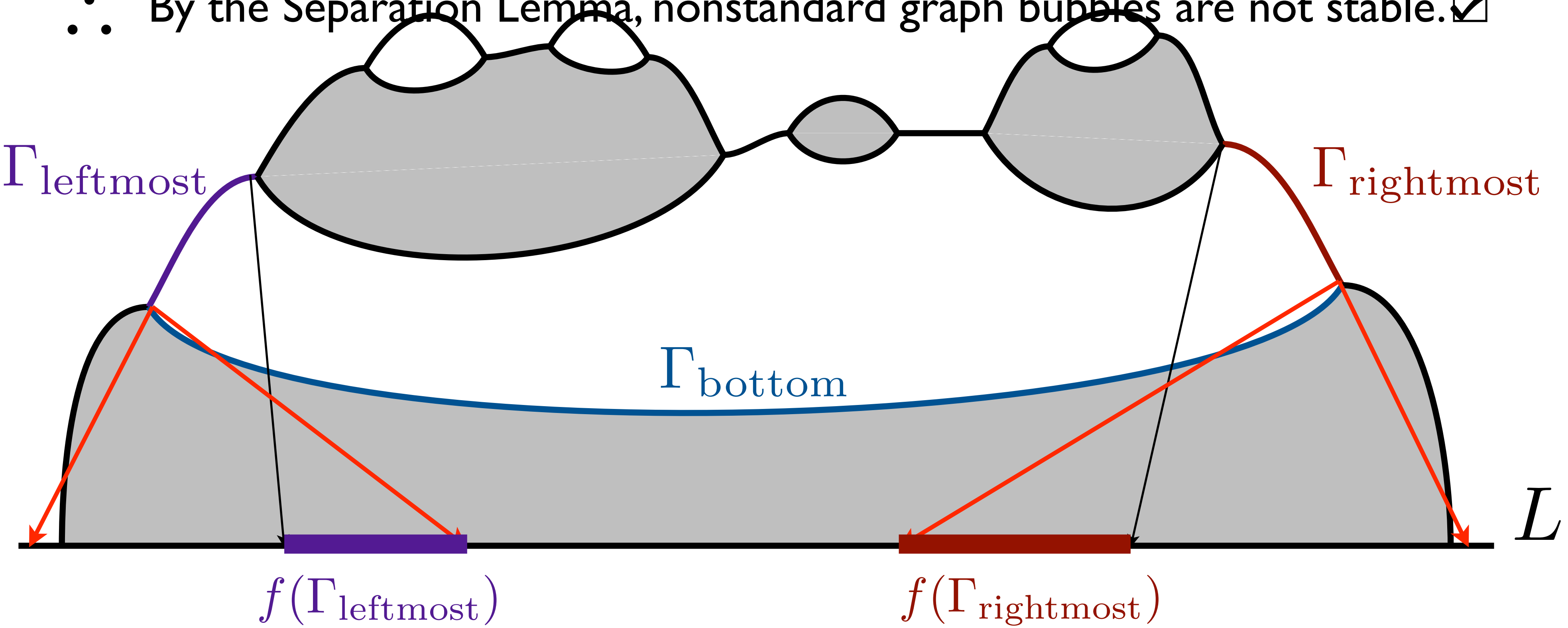


Case of graph generating curves

- **Separation Lemma:** $\{f^{-1}(x)\}$ cannot separate the generating curves
- Assume that all pieces of the generating curves are graph above L (no piece turns past the vertical)—want to find a "separating set"
- Repeating leaf argument… get $f(\Gamma_{leftmost}) < f(\Gamma_{rightmost})$
- But $f(\Gamma_{bottom})$ starts left of sup $f(\Gamma_{leftmost})$ and ends above inf $f(\Gamma_{rightmost})$

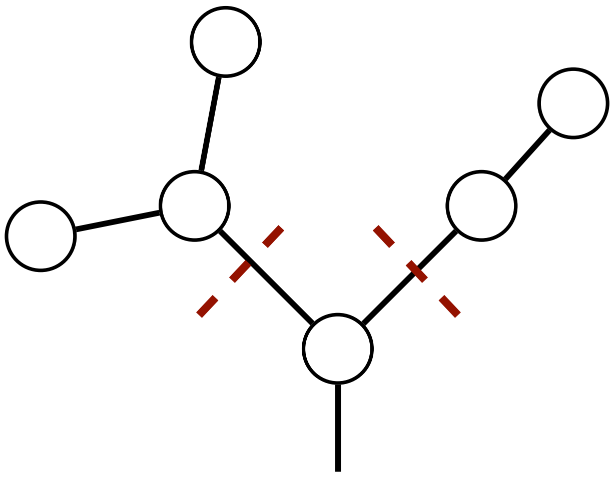
∴ There is a Γ_{bottom}, $\Gamma_{leftmost}$ separating set! $(f(\Gamma_{bottom}) \cap f(\Gamma_{leftmost}) \neq \emptyset)$

∴ By the Separation Lemma, nonstandard graph bubbles are not stable.☑

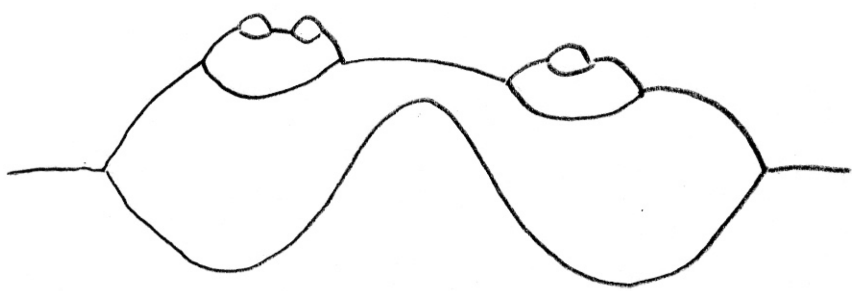


Proof sketch

- Instability by separation [HMRR '02]
- Elimination of graph nonstandard bubbles
- Inductive reduction to graph case
  - Starting at the leaves, and moving toward the root of the component stack, show that generating curves must be graph above L



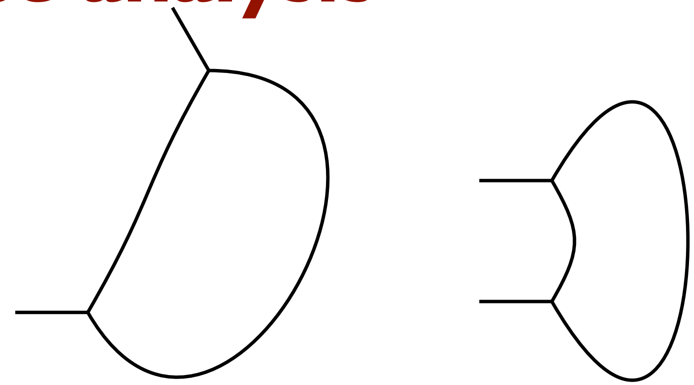
Tree

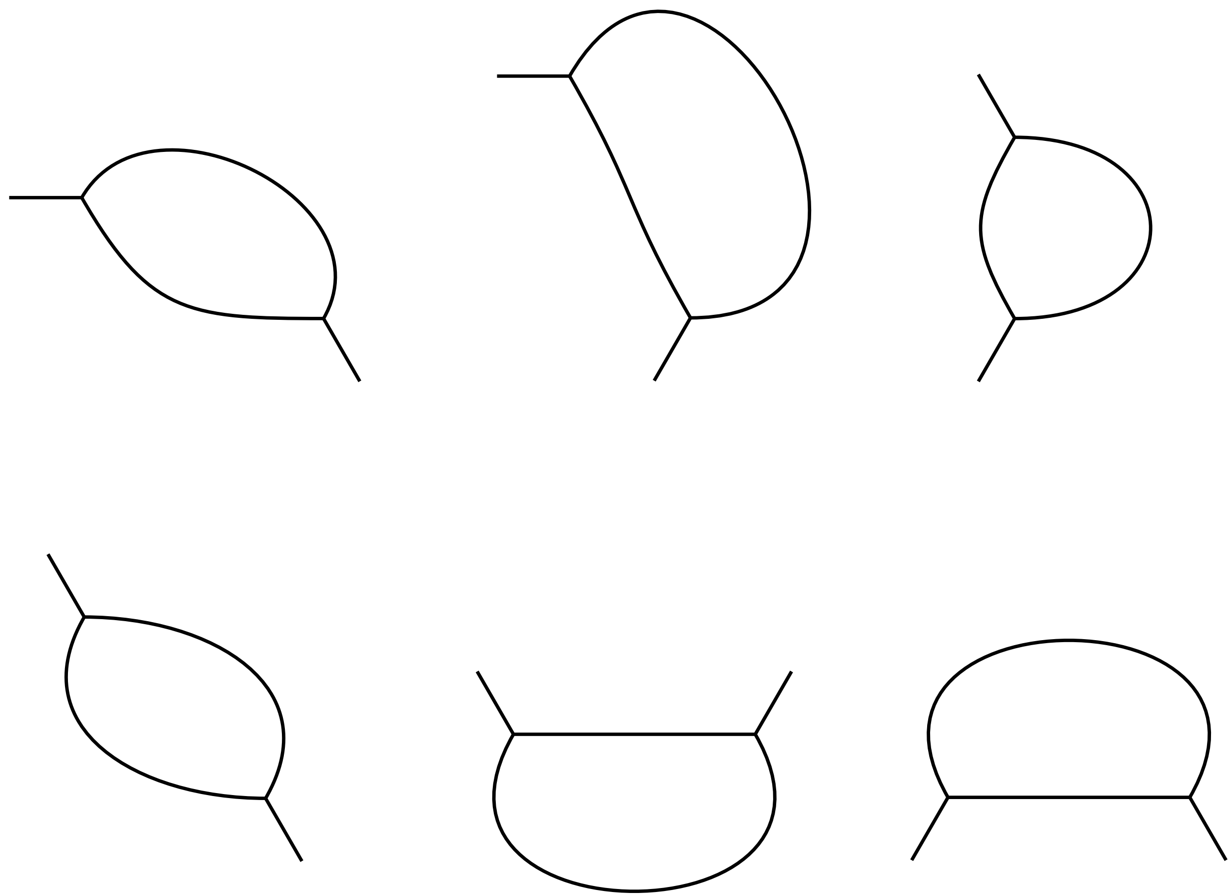


Generating curves

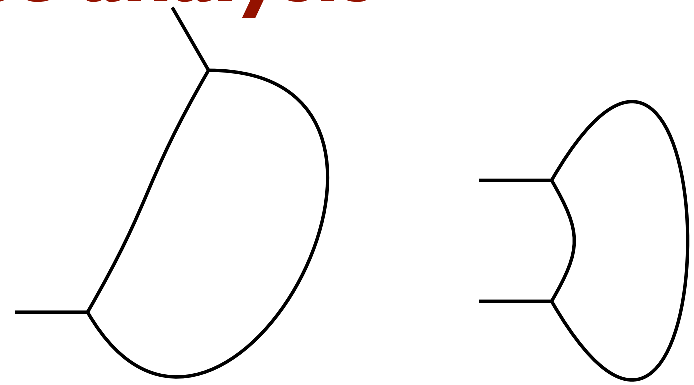
Case analysis

- Base case: Need to eliminate 8 non-graph leaf component configurations

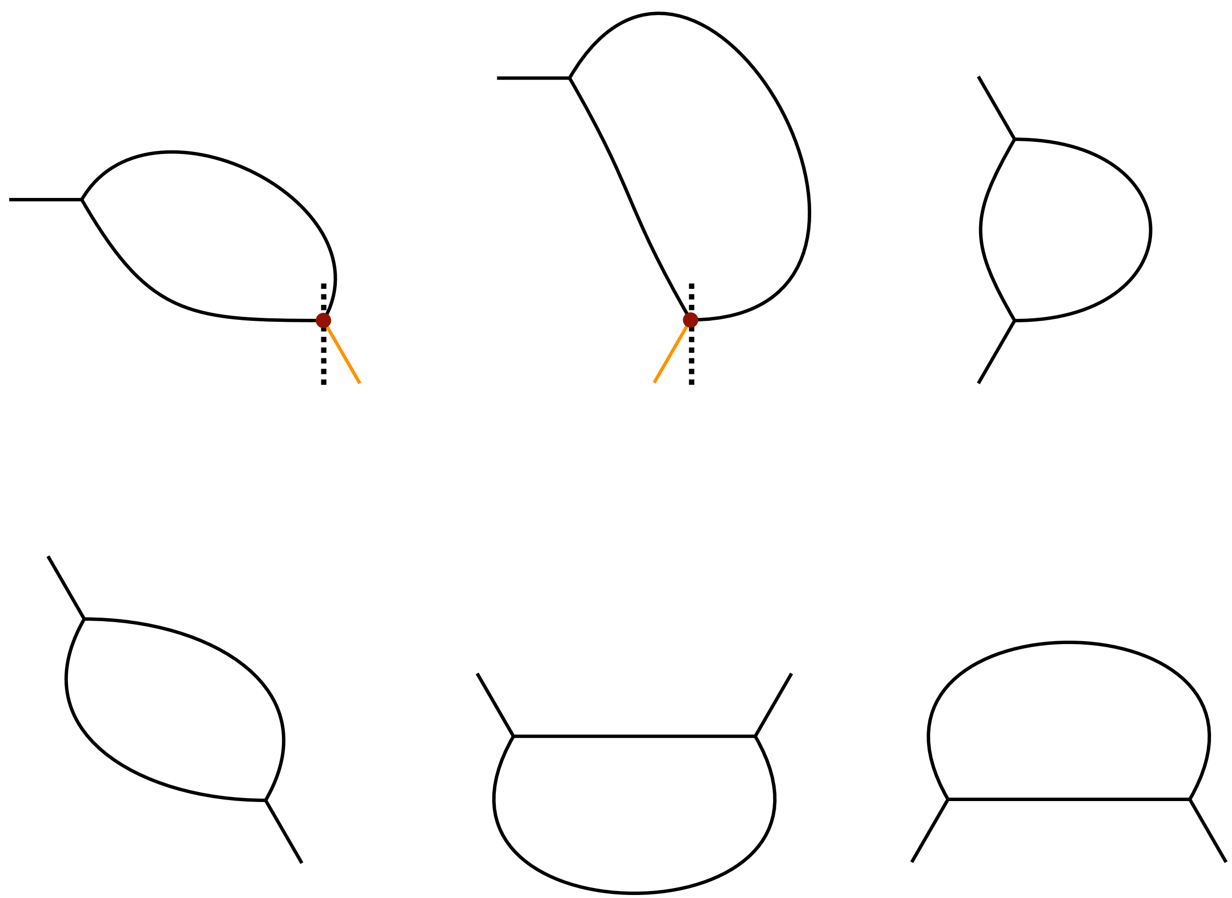




Case analysis

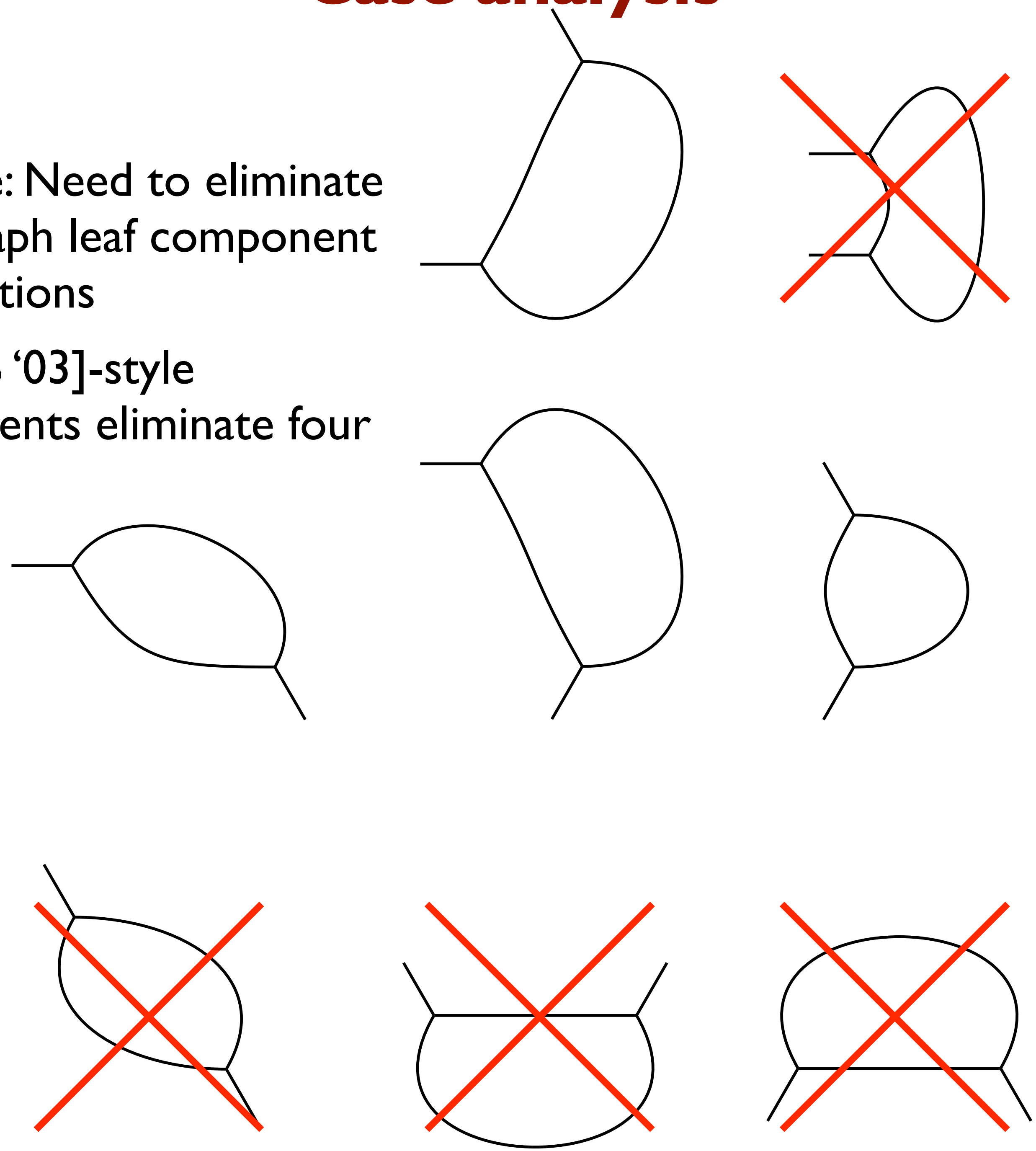


- Base case: Need to eliminate 8 non-graph leaf component configurations
  - (divided by vertex angles)



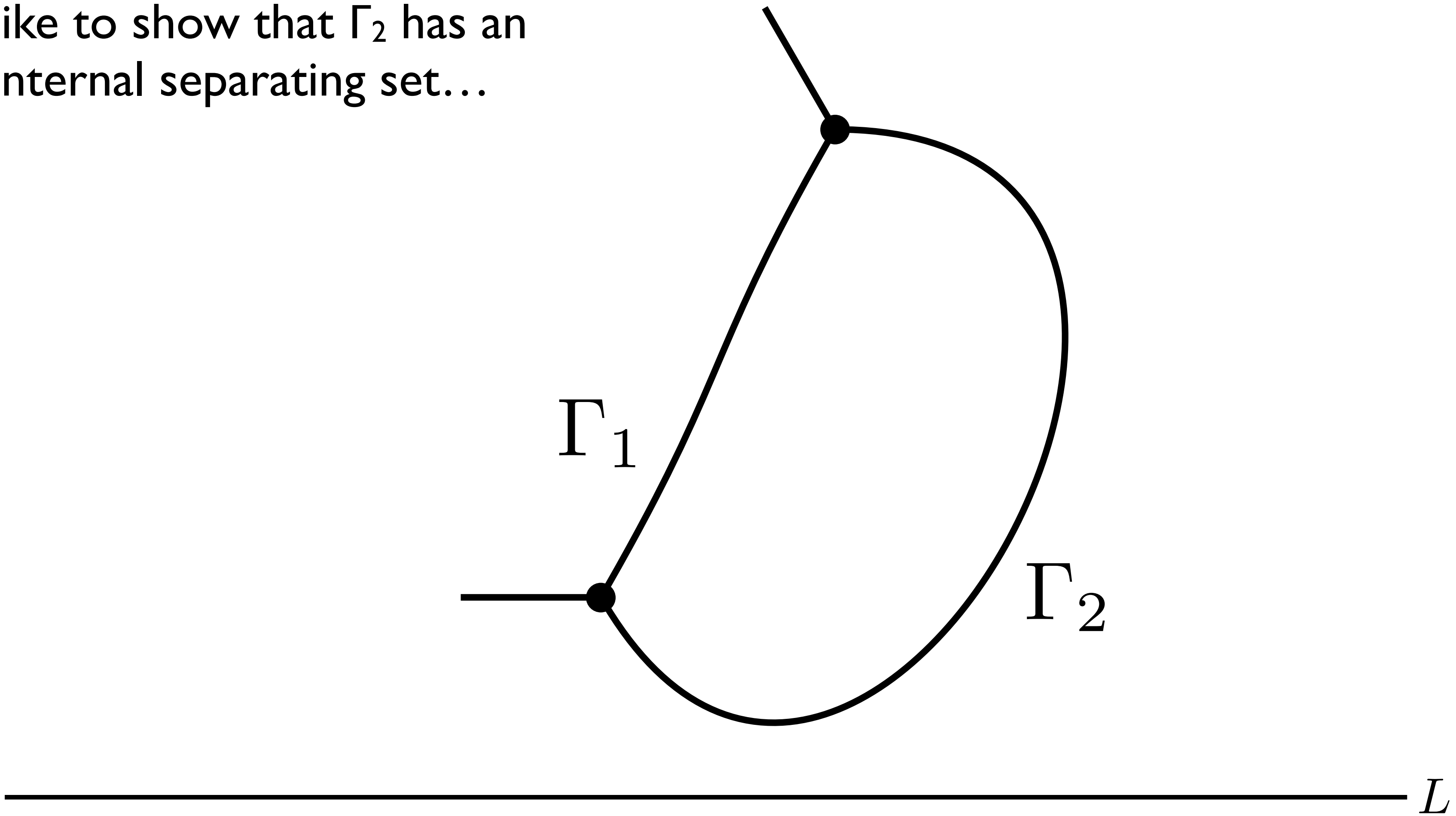
Case analysis

- Base case: Need to eliminate 8 non-graph leaf component configurations
  - [RHLS '03]-style arguments eliminate four cases



Case "(0,2)"

- To eliminate this case, we'd like to show that Γ_2 has an internal separating set…



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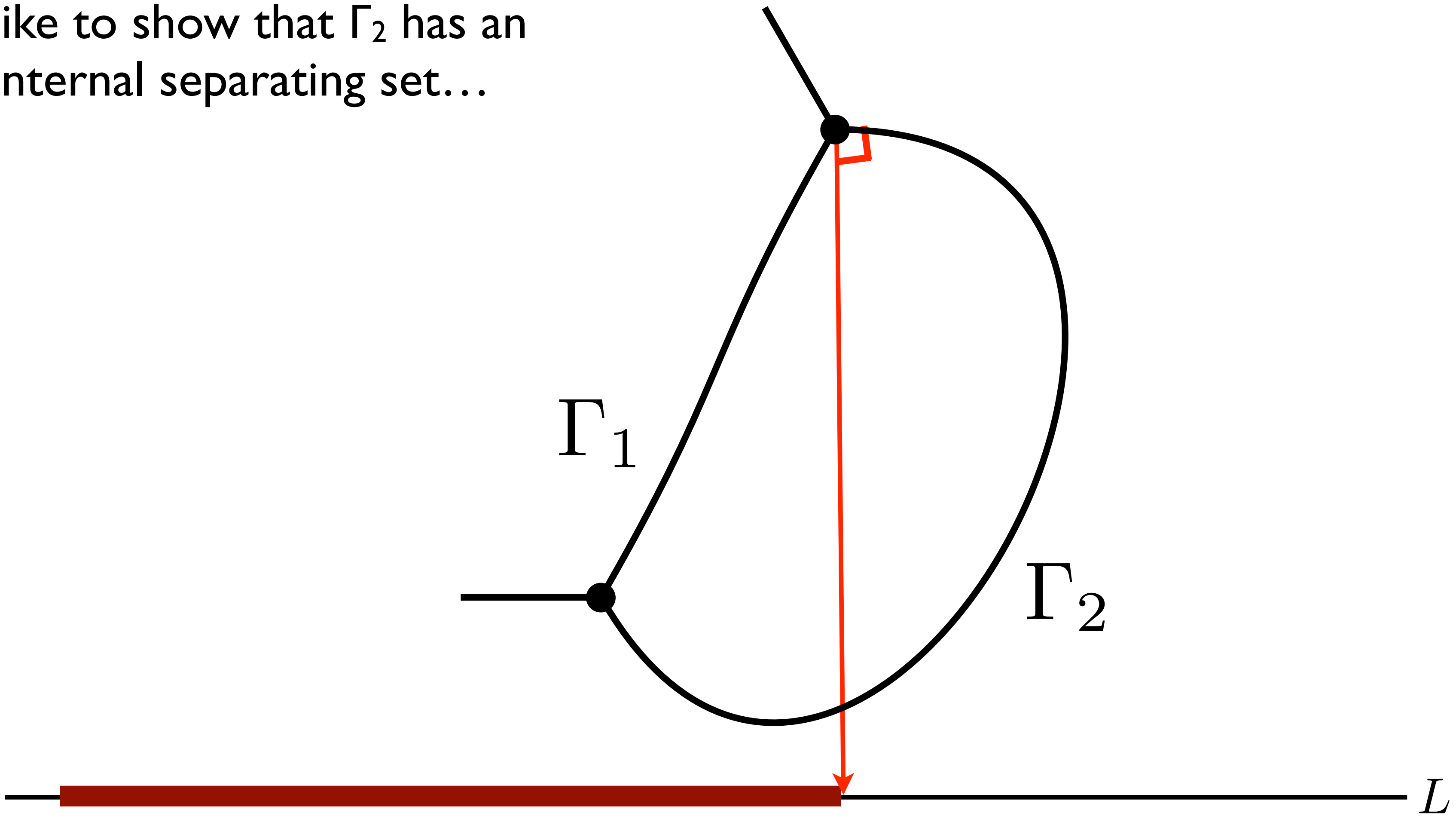

Case “(0,2)”

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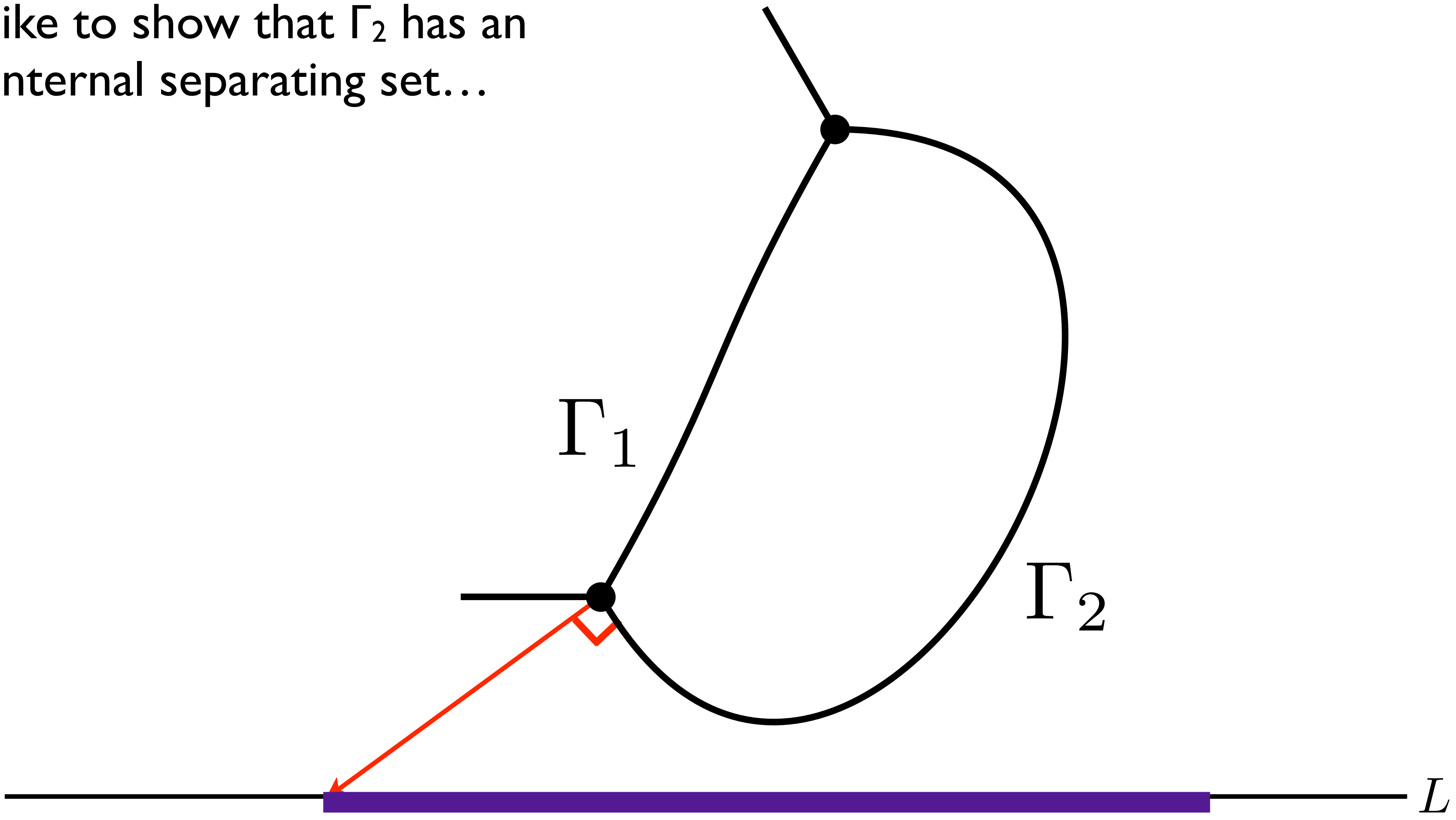

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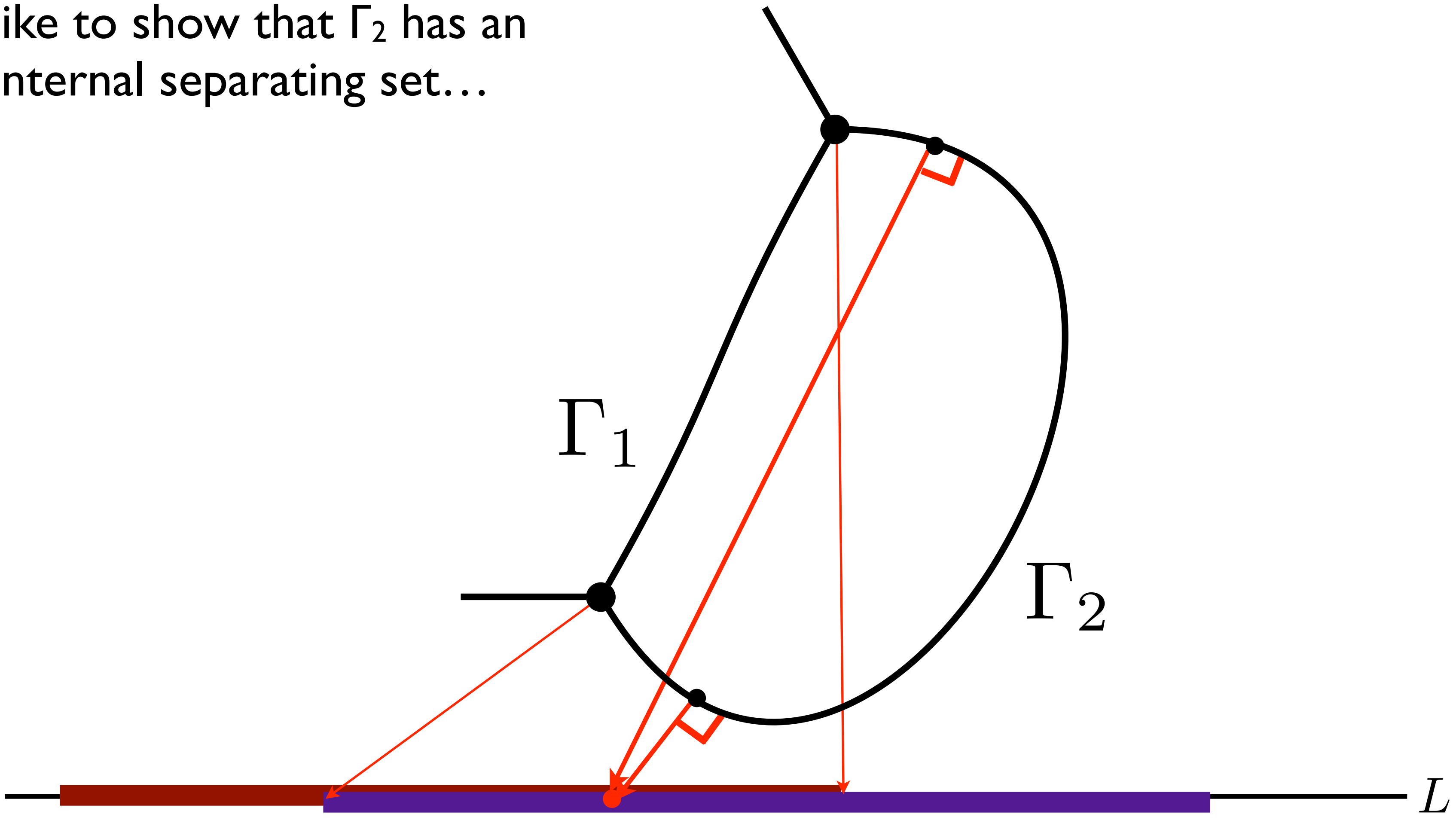
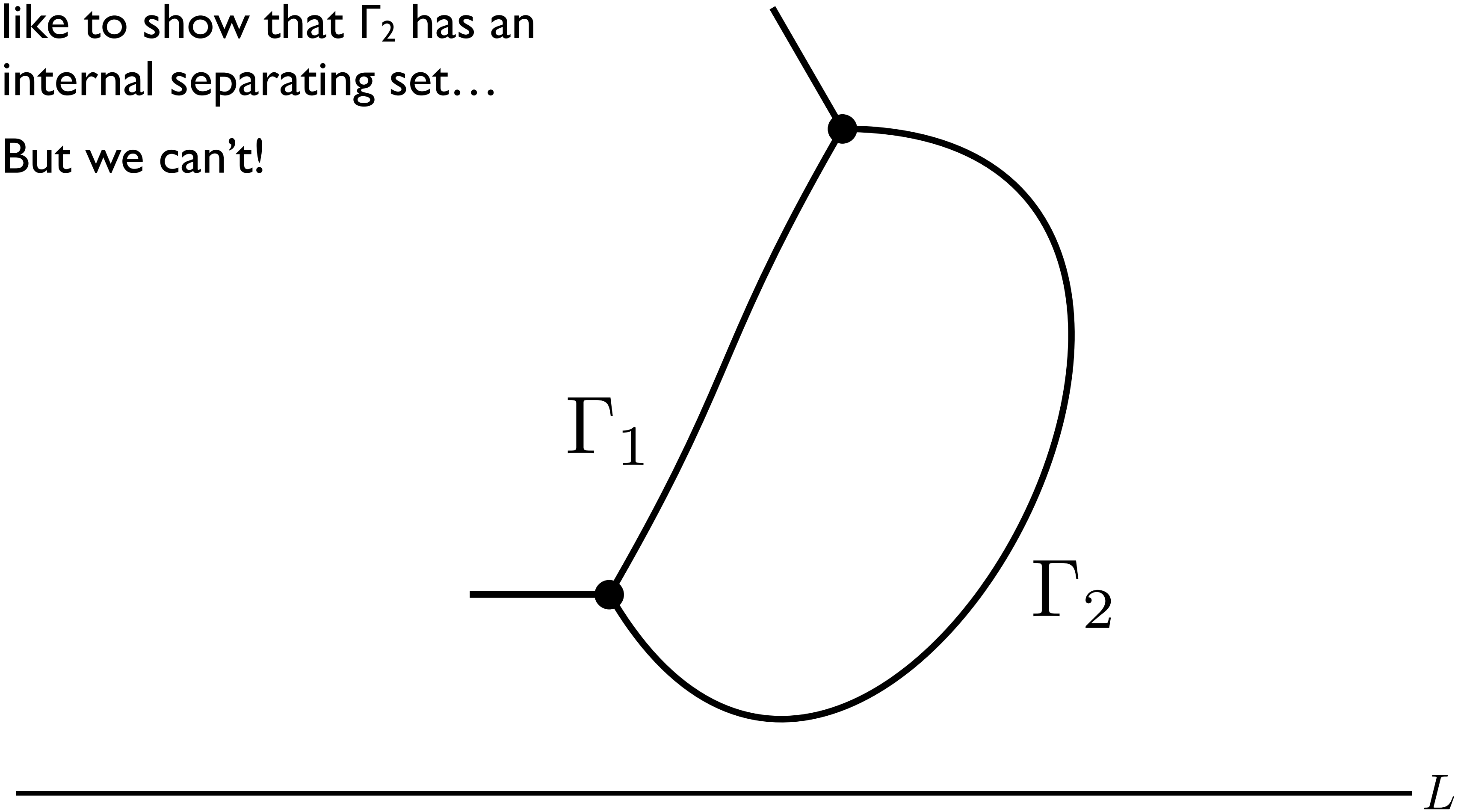


Case "(0,2)"

- To eliminate this case, we'd like to show that Γ_2 has an internal separating set…
- But we can't!



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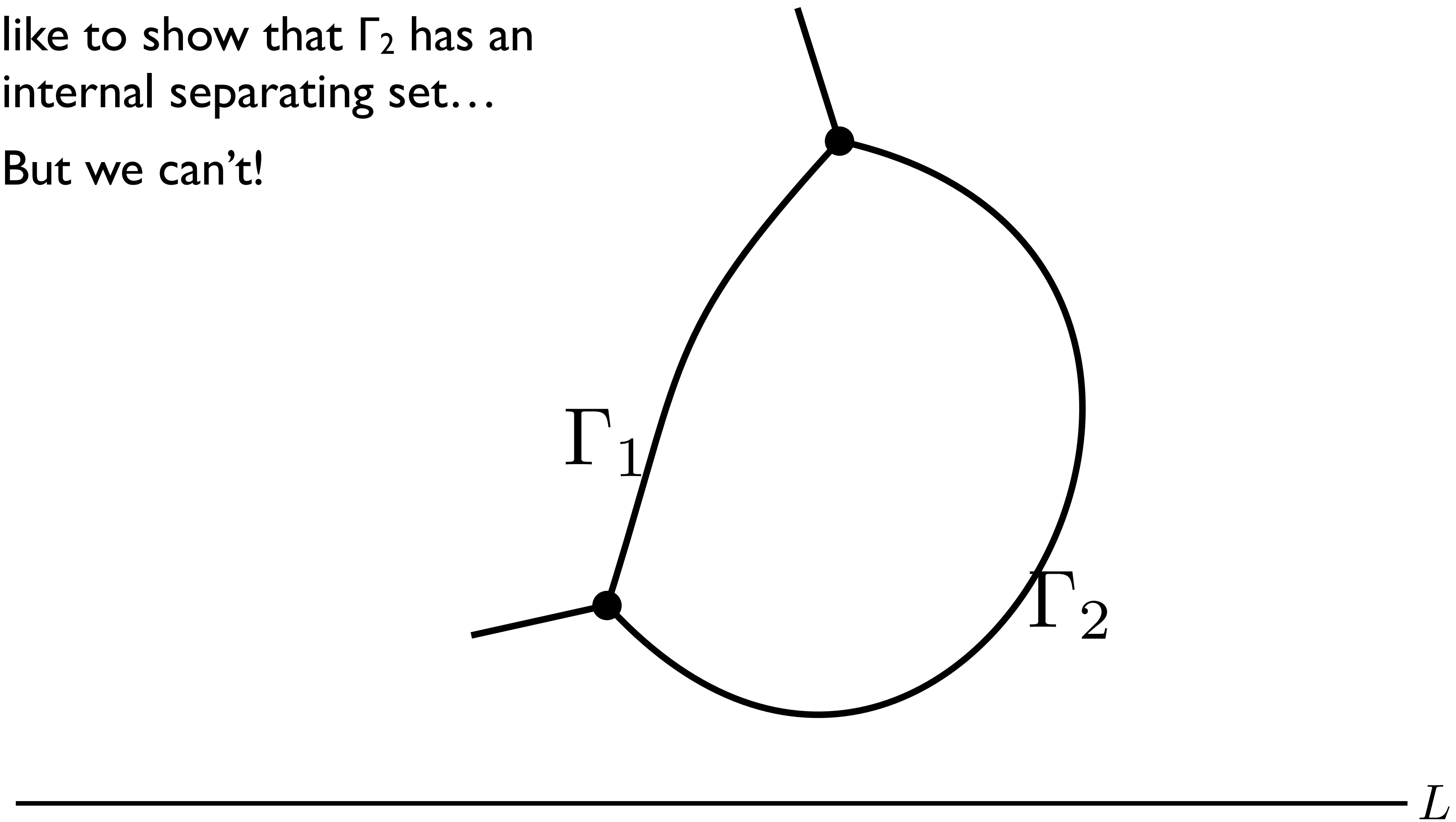

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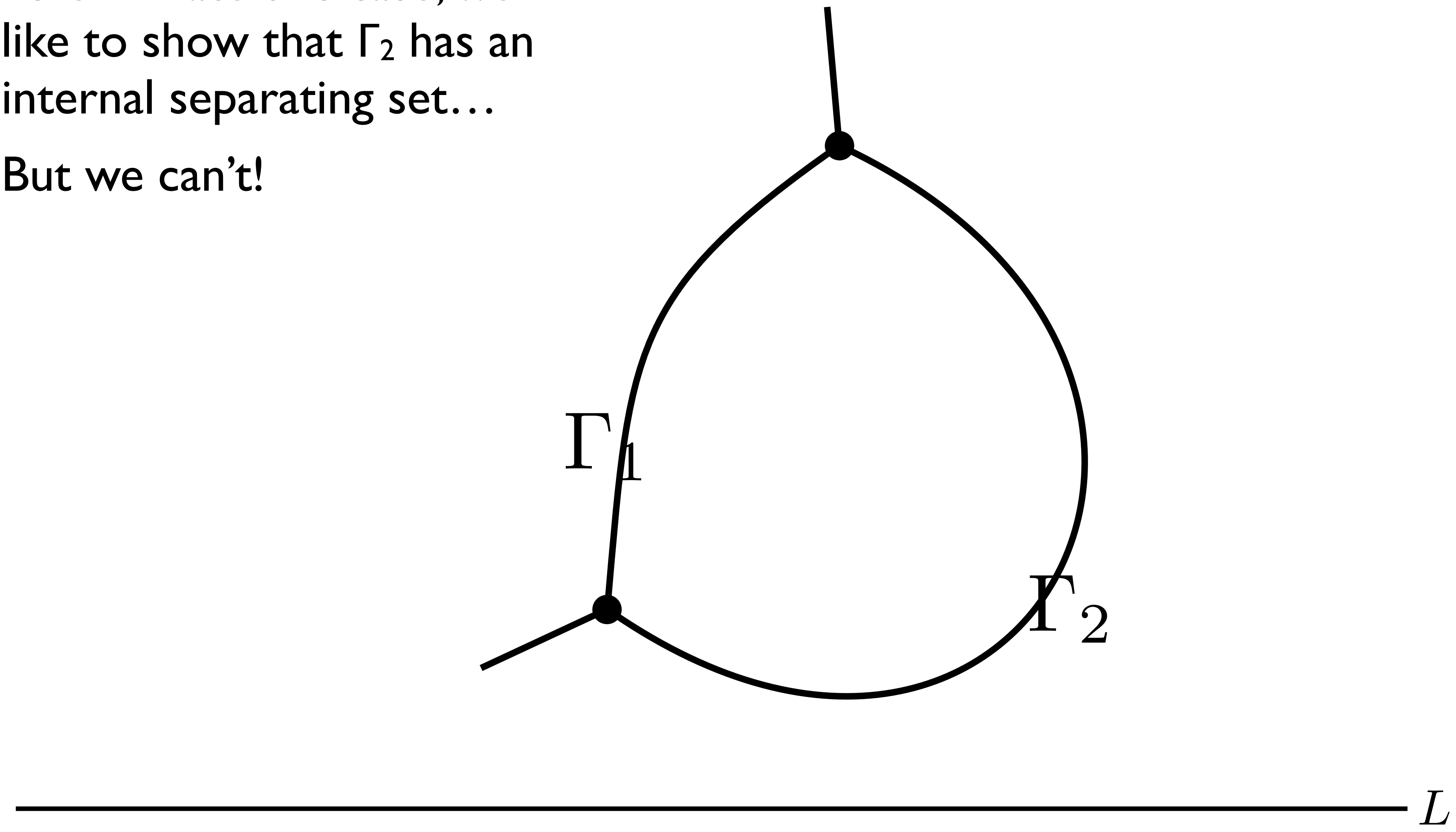

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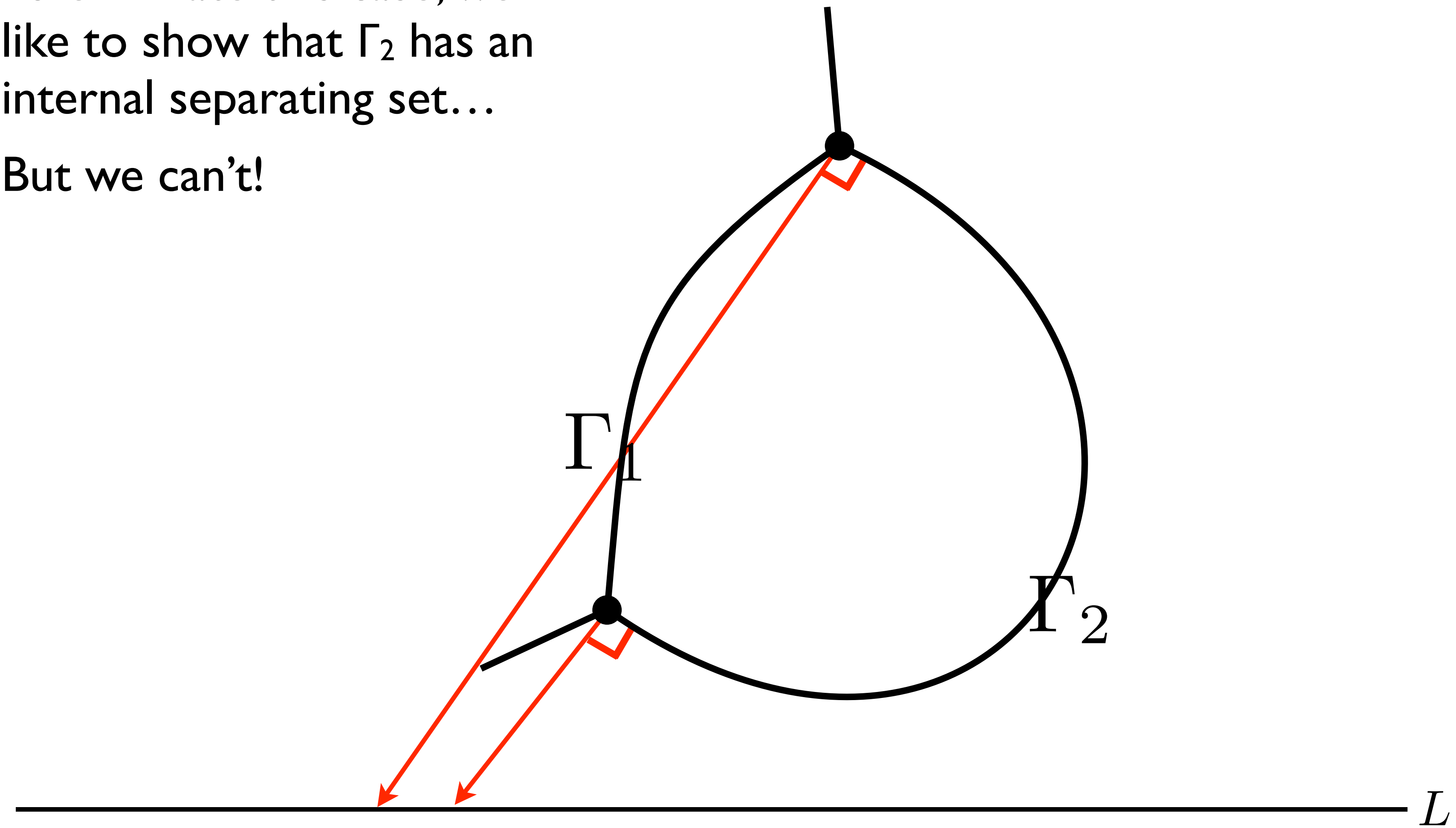
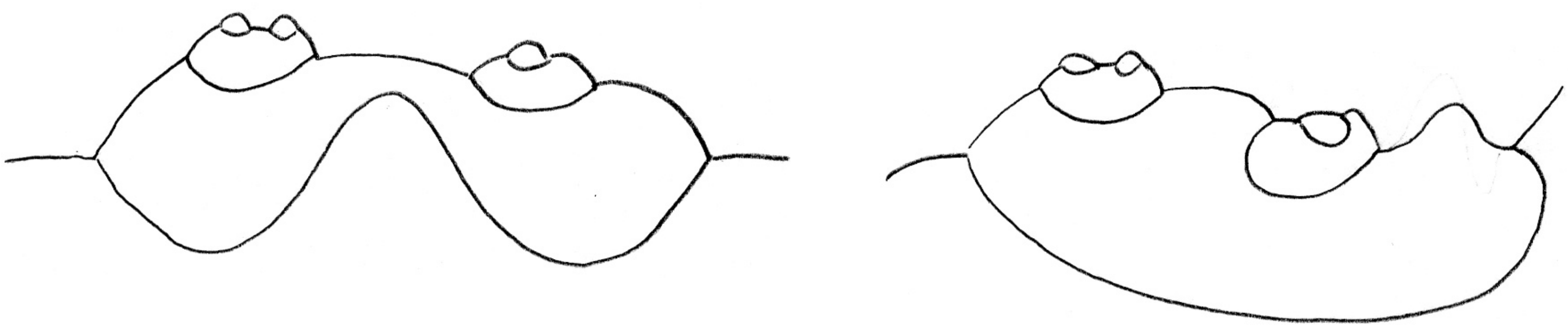


Proof sketch

- Instability by separation [HMRR ‘02]
- Elimination of graph nonstandard bubbles
- Inductive reduction to (near) graph case
  - Starting at the leaves, and moving toward the root of the component stack, show that generating curves must be (near) graph above L



- But this doesn’t work! Arguments of [RHLS ‘03] alone—eliminating “1+k” bubbles—do not suffice to eliminate “j+k” bubbles. Need to know more about the generating curves…

Hutchings Structure Theorem

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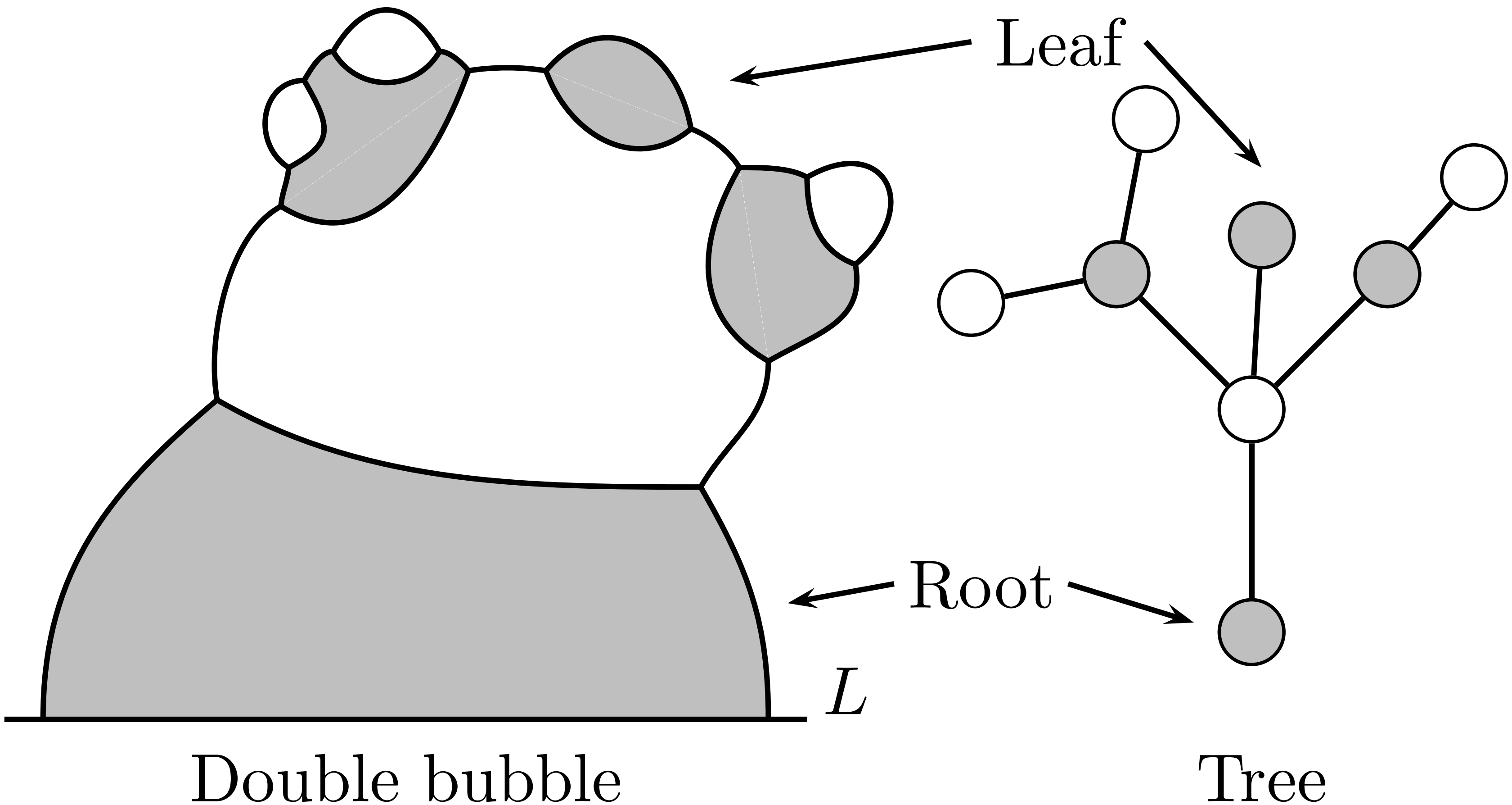
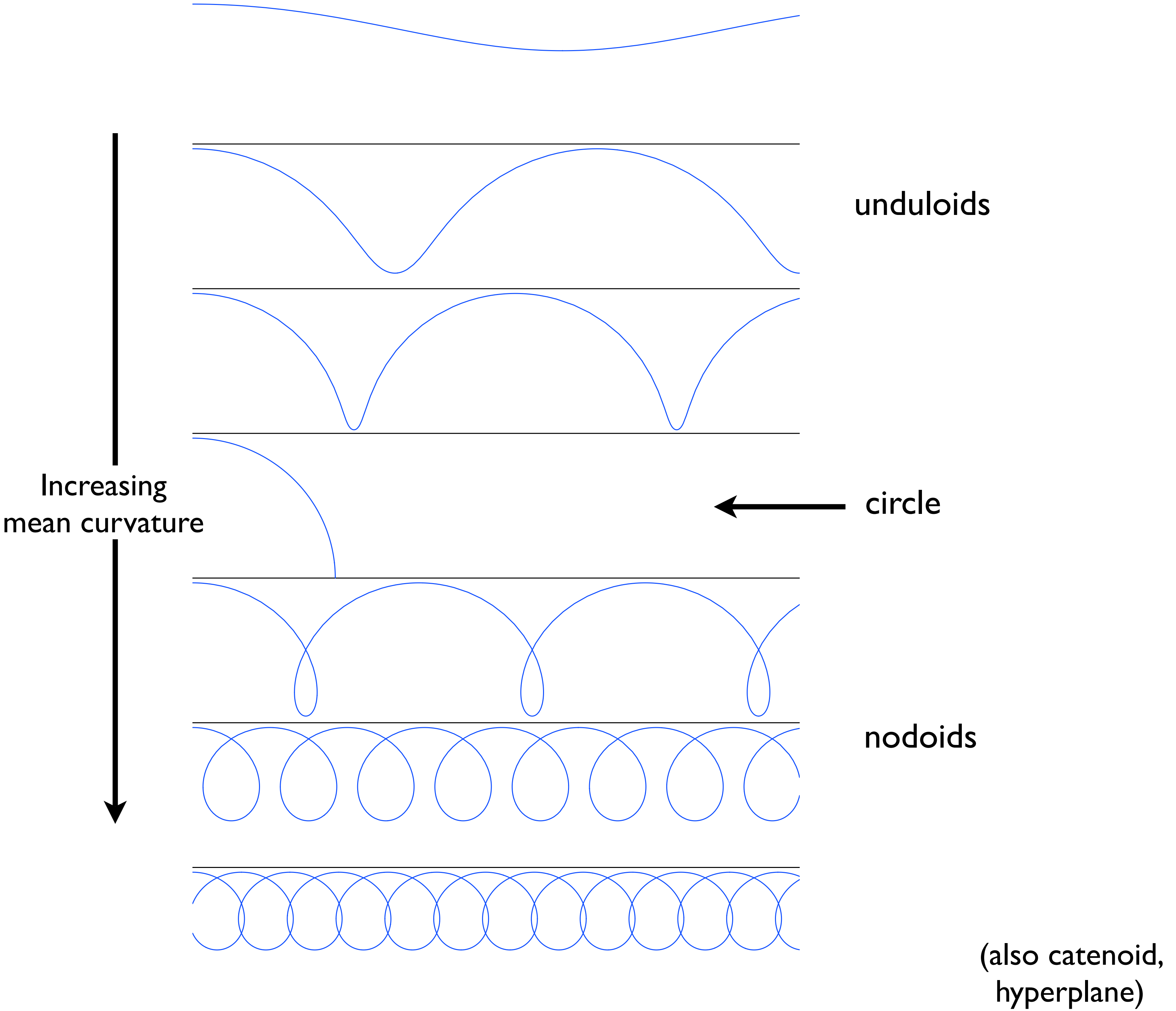
  Boundaries are constant-mean-curvature surfaces meeting at 120° angles.

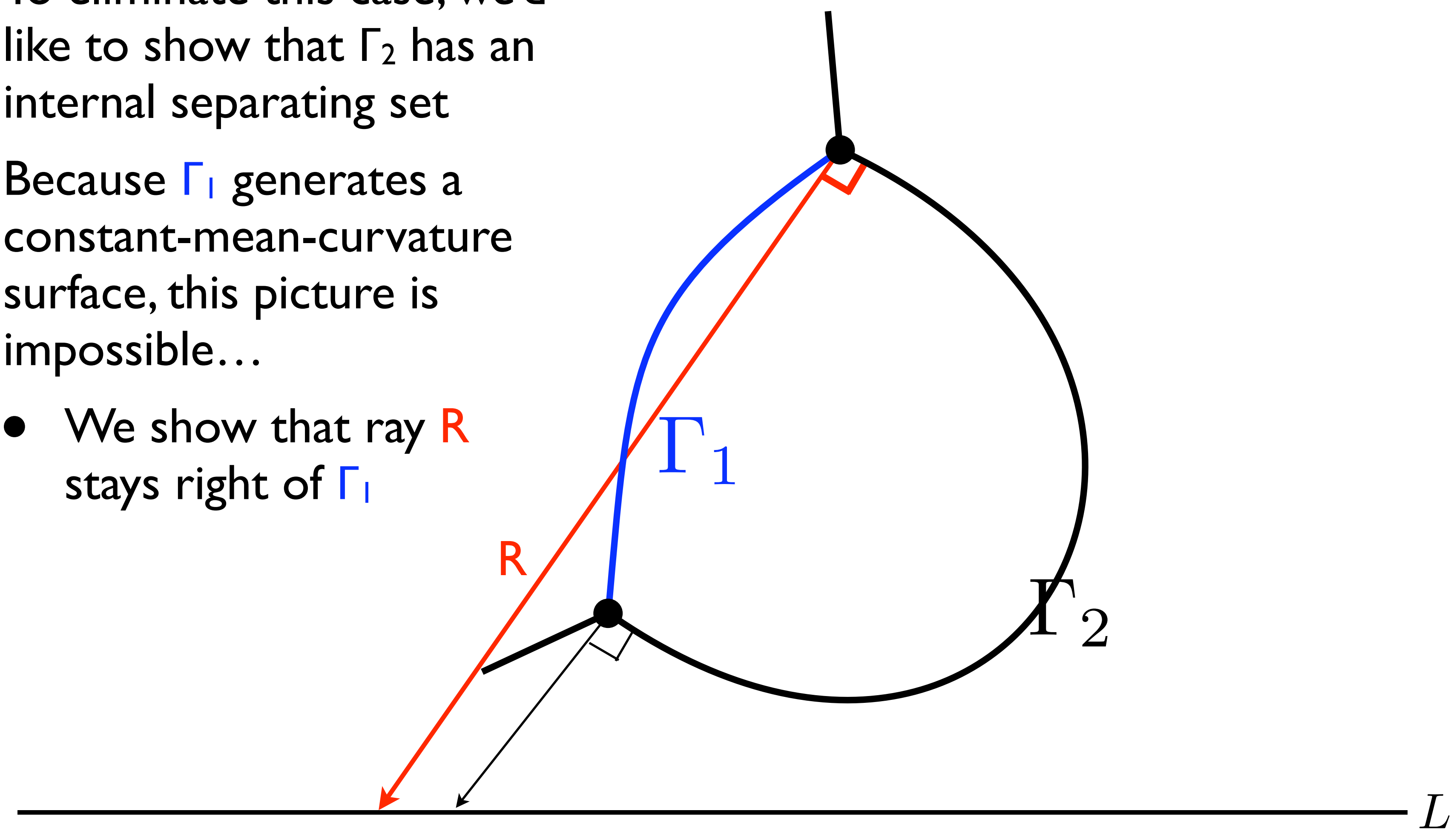


Constant-mean-curvature surfaces of revolution



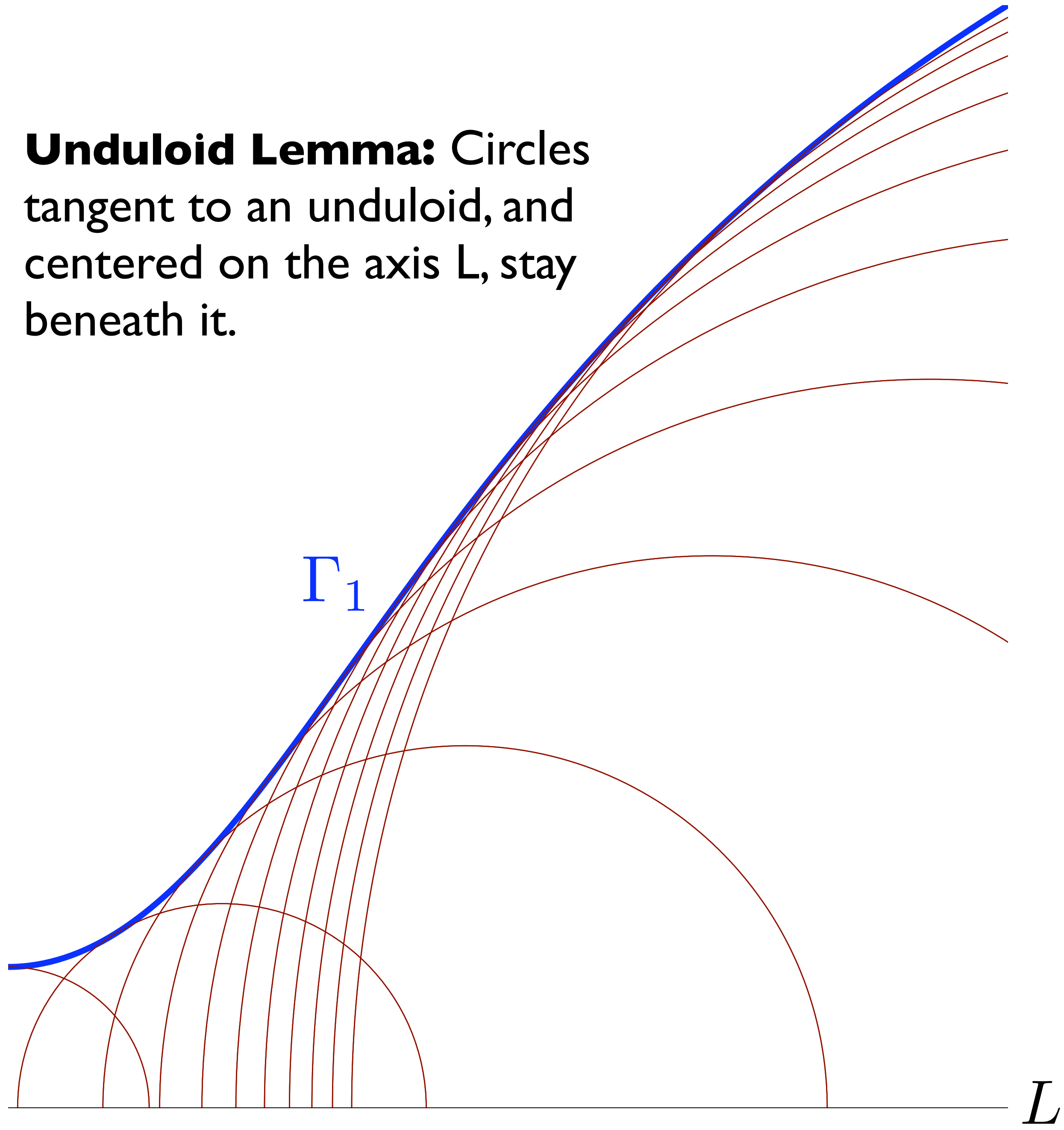
Case "(0,2)"

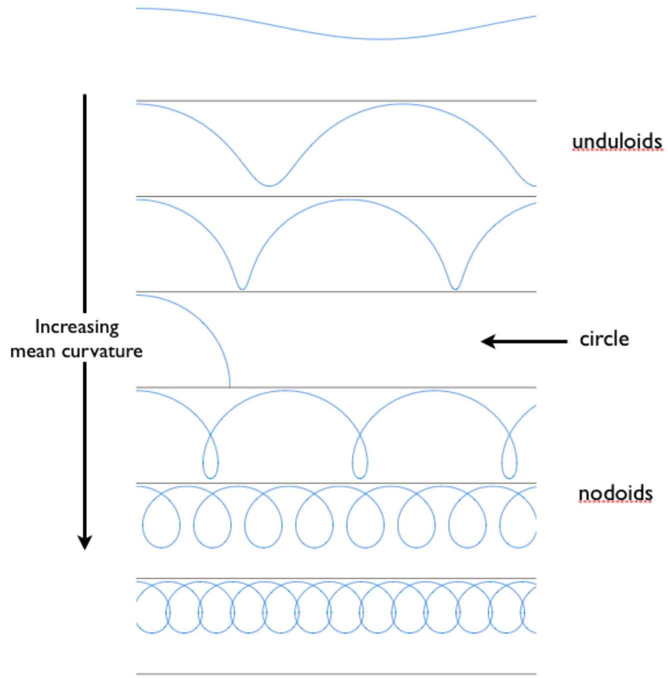
- To eliminate this case, we'd like to show that Γ_2 has an internal separating set
- Because Γ_1 generates a constant-mean-curvature surface, this picture is impossible…
  - We show that ray R stays right of Γ_1



If Γ_1 is an unduloid:

- **Unduloid Lemma:** Circles tangent to an unduloid, and centered on the axis L, stay beneath it.





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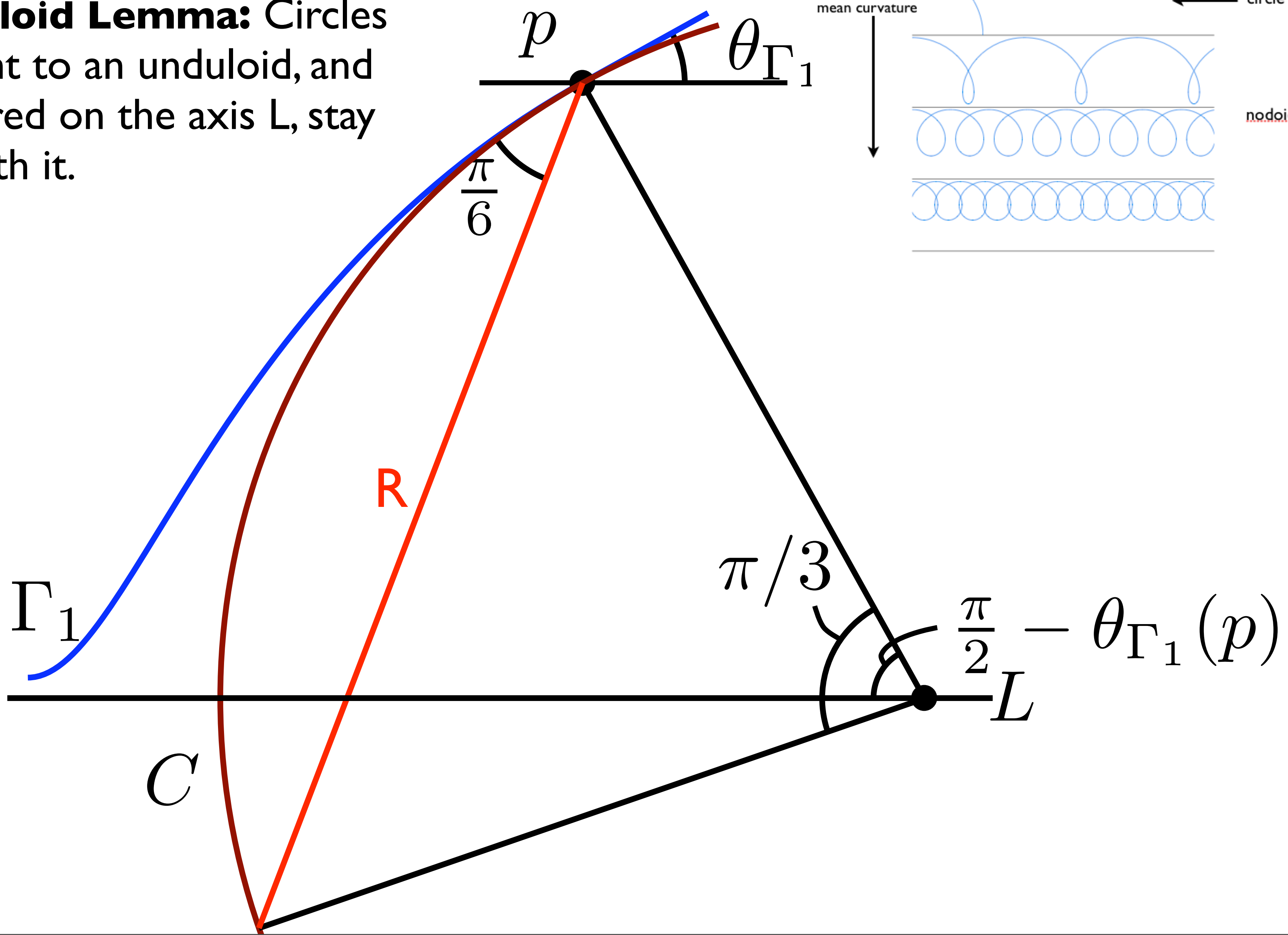
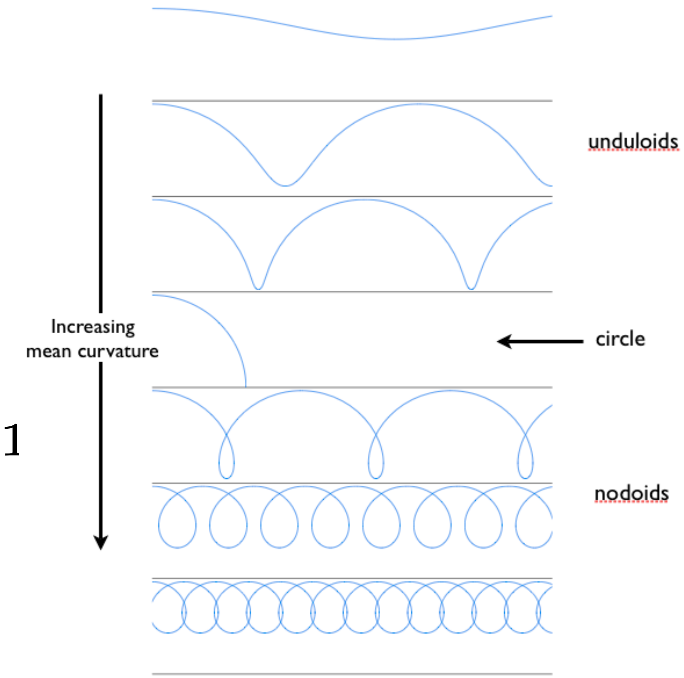
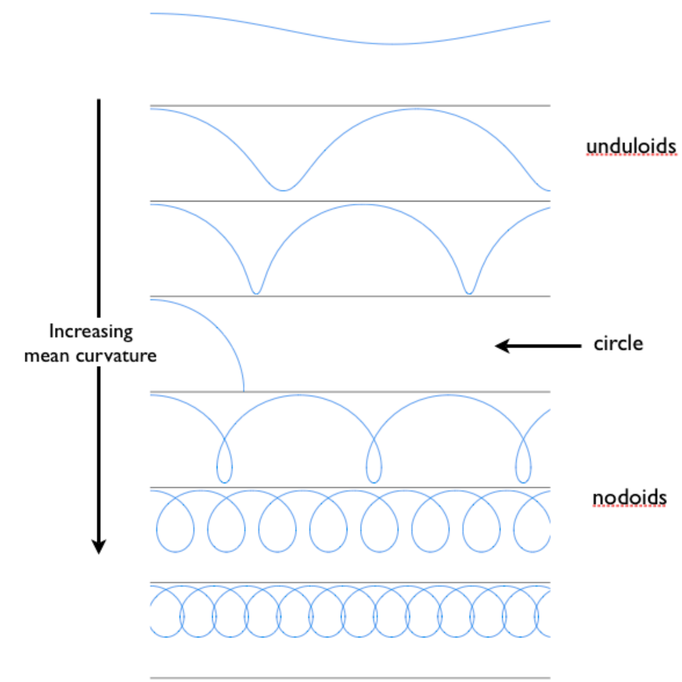
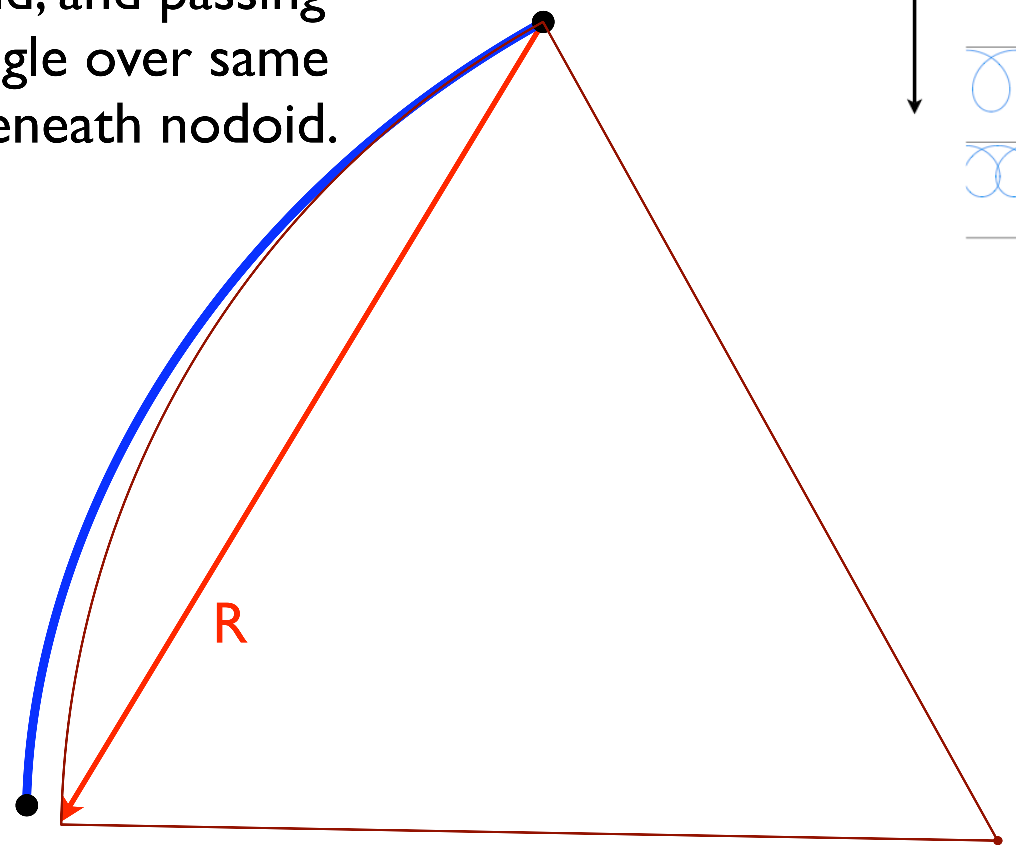




If Γ_I is a nodoid:



- **Nodoid Lemma:** Circles tangent to nodoid, and passing through same angle over same arclength, stay beneath nodoid.



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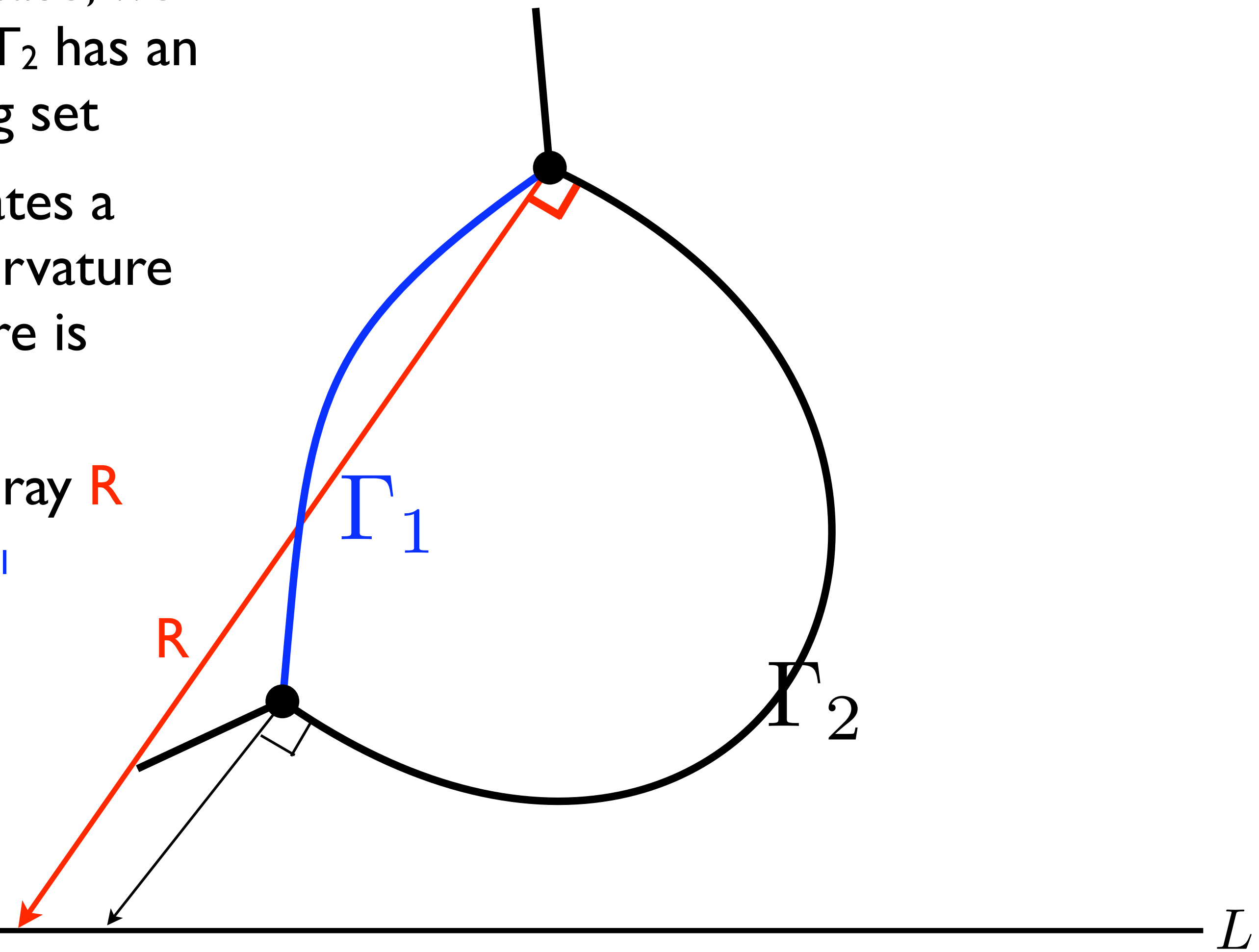
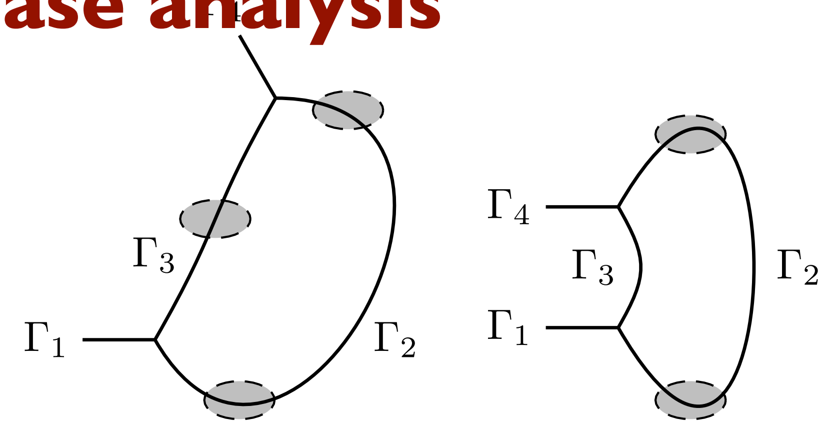
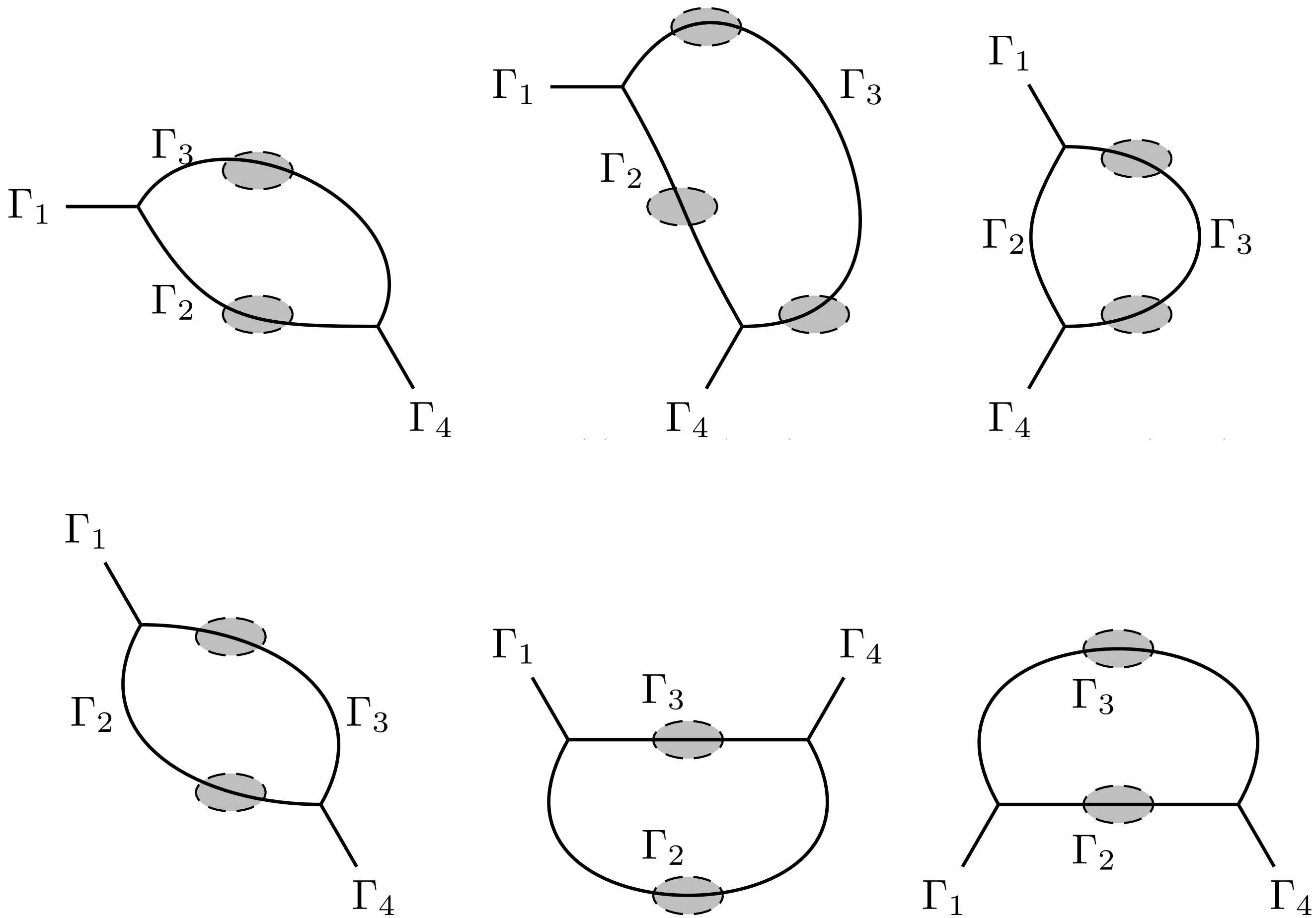


Case analysis

- Need to eliminate 8 non-graph component configurations





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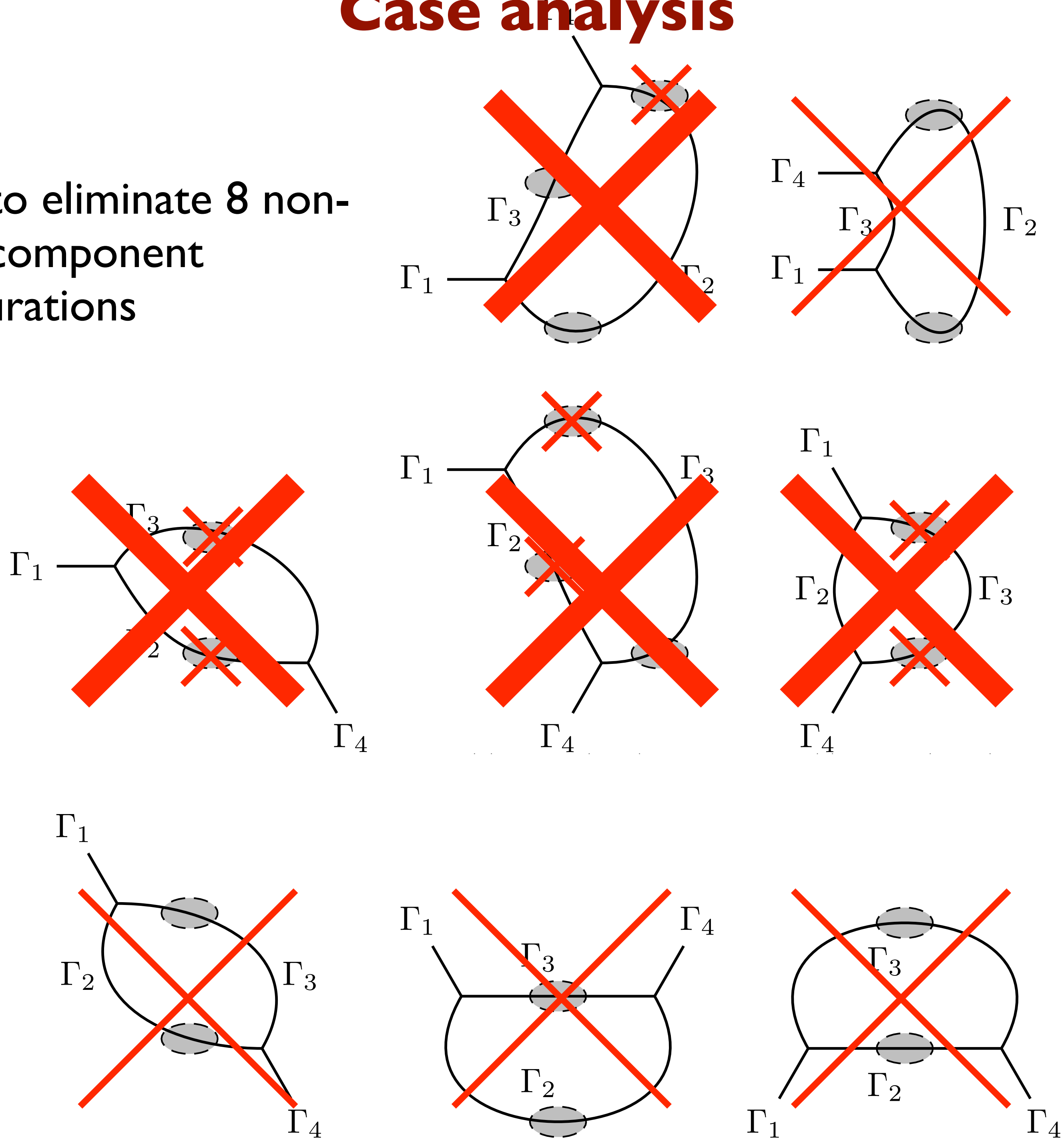
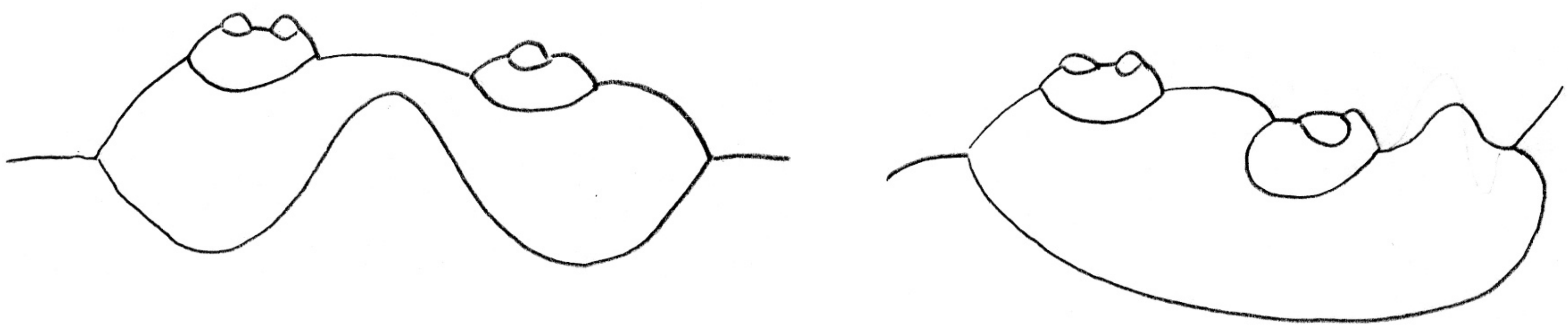


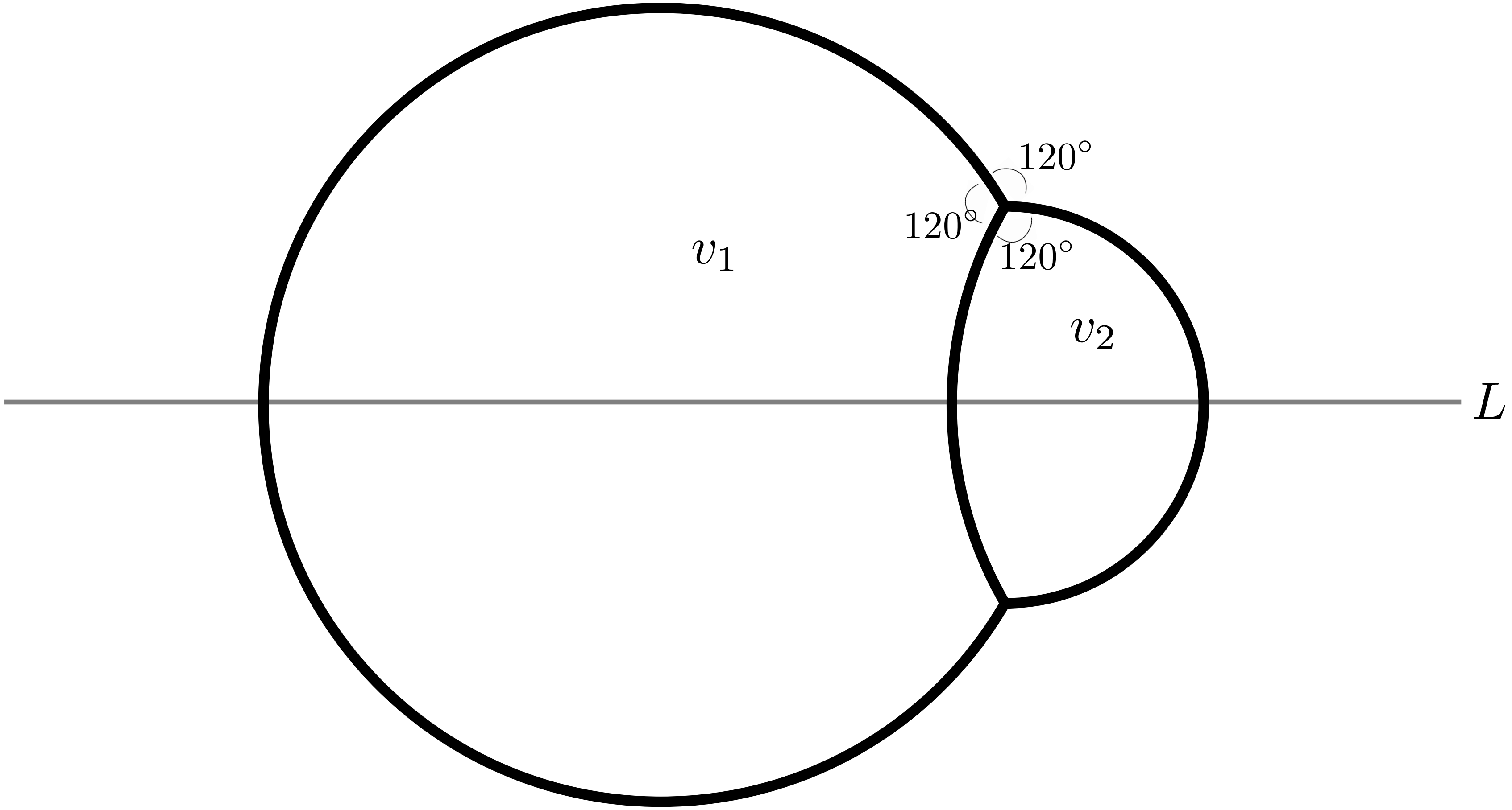
Proof sketch

- Instability by separation [HMRR '02]
- Elimination of graph nonstandard bubbles
- Inductive reduction to (near) graph case
  - Starting at the leaves, and moving toward the root of the component stack, show that generating curves must be (near) graph above L



Double Bubble Theorem

- **Theorem:** The least-area way to enclose and separate two given volumes in $\mathbf{R^n}$ is the standard double bubble.



- three spherical caps centered on the axis L, meeting at 120 degree angles